Shaft Deflection—A Very, Very Long Example

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Abstract – Most textbooks in mechanics of materials and components of machine describe numerous methods for determining shaft (beam) deflections. The examples are quite varied, but most are not interrelated with follow-up examples in which the same problem is attacked using different methods. Further, most examples are worked out by hand and do not emphasize numerical methods used in common practice. Here, the deflection of a stepped shaft in bending was solved using the following methods: closed-form successive integration of the bending moment equations, Castigliano's second theorem, numerical successive integration using the trapezoid rule, finite element method, and estimation by beam table superposition. Computational tools, such as MATLAB, Excel, and Maple were used in the solution process. The advantages and disadvantages of the methods used are discussed as well as observations from use of the example.

Keywords: beam deflection, Castigliano's second theorem, double integration, finite element modeling, numerical integration

INTRODUCTION

The basic idea of this paper is to present, or at least, outline the solution to a shaft deflection problem using several different methods. Studying the same problem from multiple perspectives is one way to gain a better overall understanding. Different problem solving strategies can lead to the same solution and allow a student to see how useful one method is when compared to another. Two other benefits from using different methods to study the same problem are confidence building and estimation-skill development. Certainly, if students have successfully arrived at one solution to a problem, they are likely to be more confident when trying another method. Also, having worked through the several different methods, students can begin to judge situations where one solution method might be preferred over others. Sometimes, the other method or methods may be not as accurate or as useful, but they can provide enough accuracy for estimating the solution. Students need to practice the art of estimation so that they can validate their own solutions.

In this paper, we will consider only the statically determinate problem. However, the same methods could be applied to a statically indeterminate problem. Also, bending is restricted to a single plane in this paper.

There are many different solution methods for beam/shaft deflections. Integration schemes—both direct successive integration and graphical integration—are the primary methods taught in mechanics of materials. Energy methods are very powerful, but often have less emphasis in the first mechanics of materials course. The general advice we give in machine design practice is to superimpose solutions from beam tables if the beam or shaft has a constant cross section. When a stepped shaft is to be studied, direct successive integration becomes tedious. For a stepped shaft, we recommend an energy method, such as Castigliano's second theorem, if the deflection or slope is required at only a few locations. Otherwise, we recommend numerical methods, such as successive numerical integration or finite elements. However, we always recommend using beam table superposition for bounding estimates.

There are many mechanics of materials texts available to students. Timoshenko [1] and Popov [2] are the classics in the field. Beer [3], Higdon [4] and Riley [5] are long established. Philpot [6], Vable [7] and Allen [8] are newer treatments. These texts provide many excellent examples of both statically determinate and indeterminate beam

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problems. A variety of methods are used. For example, Hibbeler's [9] chapter "Deflections of Beams and Shafts" includes successive integration of the load intensity and successive integration of the bending moment, singularity (Macaulay) functions, the moment-area method, and the principle of superposition. In a later chapter, "Energy Methods," virtual work and Castigliano's second theorem are used to determine deflections.

As with mechanics of materials, there are many component machine design texts. All these texts assume that students have already taken a course in mechanics of materials. Shigley[10], Phelan [11] and Spotts [12] are notable classics, while Johnson [13] provided a new approach. Newer entries include Norton [14], Ugural [15] and Collins [16]. All these texts assume that students have taken mechanics of materials. This assumption is also true of most machine design courses.

There are many different tools for solving such problems. Powerful graphing calculators often have symbolic and numerical capabilities for solving differential equations and for integration. Computer software includes specialty educational programs for mechanics of materials, small locally-written finite element codes, commercial finite element codes, computer algebra systems and related mathematical analysis programs and spreadsheets. With all these computer tools available, many faculty and students will still choose to write their code.

The Problem Statement

The chosen problem is a stepped shaft with simply supported (statically determinate) boundary conditions and two concentrated forces. This problem was taken from Juvinall and Marshek [17]. The step changes in diameter provide an opportunity to demonstrate how changes in area moment of inertia affect the solution. They also provide an opportunity to demonstrate how solutions for beams with constant cross sections can be used for estimation.

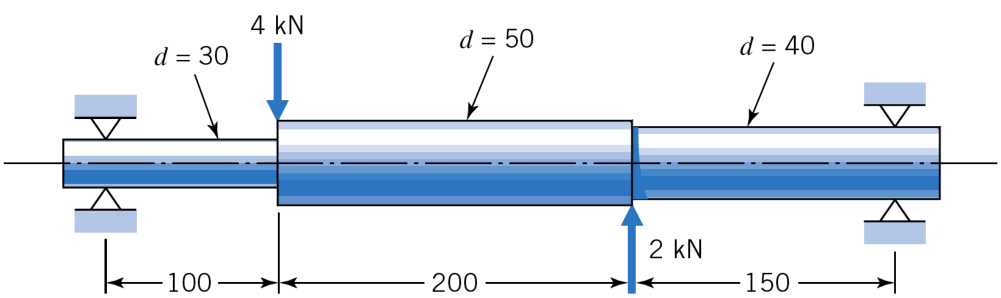

Figure 1: Stepped Shaft (all dimensions in mm) [17]

In Figure 1, the boundary conditions (double triangle symbols) are given by radial bearings. These bearings can be treated as pins or rollers depending on whether or not axial (thrust) forces are present. In this paper, we assume that that there are no axial forces so that the equilibrium equation in the x direction is identically satisfied. In Figure 2(a), a free-body diagram of the shaft is given that is consistent with this assumption. The other two equilibrium equations are

$$\sum F_y = R_1 - F_1 + F_2 + R_2 = 0 \tag{1}$$

$$\sum M_z = -F_1 a + F_2 b + R_2 L = 0 \tag{2}$$

with y taken as shown in Figure 2(a) and z taken out of the paper toward the reader. Further, the moment is taken at the left end in a counterclockwise direction. The solution for the reactions in terms of applied forces is

$$R_1 = \frac{L-a}{L}F_1 - \frac{L-b}{L}F_2 = \frac{7}{9}F_1 - \frac{1}{3}F_2 = 2444 \text{ N} \tag{3}$$

$$R_2 = \frac{a}{L}F_1 - \frac{b}{L}F_2 = \frac{2}{9}F_1 - \frac{2}{3}F_2 = -444 \text{ N.} \tag{4}$$

In Figure 2(b), the free-body diagram for the section $0 < x < a$ is given. The moments must be developed for each cross section. Figure 2(c) and Figure 2(d) show the free-body diagram for the sections given by $a < x < b$ and

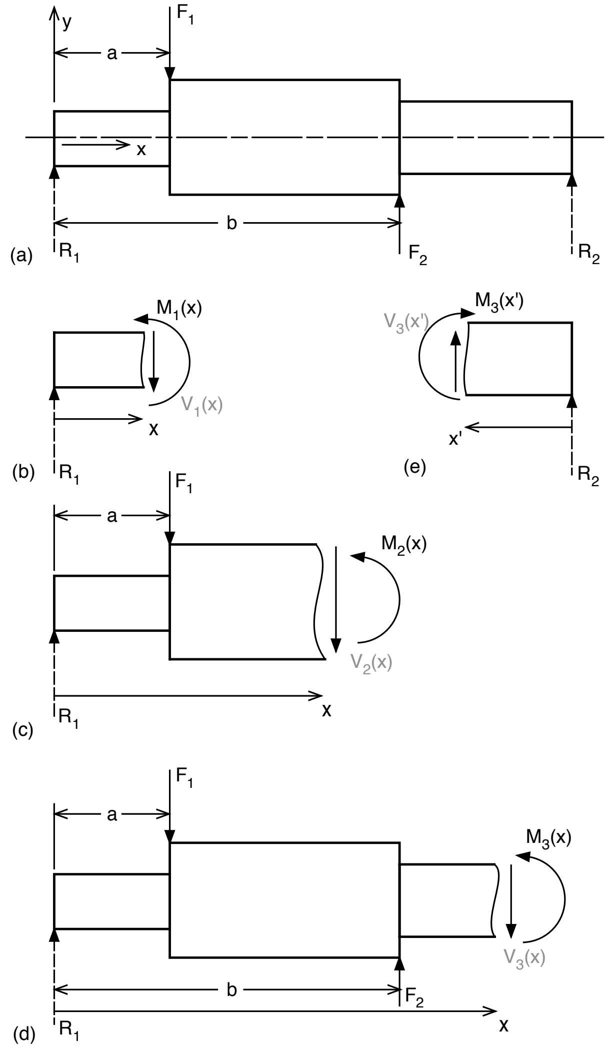

Figure 2: Free-Body Diagrams

$b < x < L$, respectively. As an alternative to using Figure 2(d), a free-body diagram is drawn in Figure 2(e) with the x origin given at the right end of the shaft (x'). The equations for equilibrium can be written for each free-body diagram to determine shear force $V(x)$ and moment $M(x)$ on each section. For brevity, only the results for $M(x)$ are given here as

$$M_1(x) = R_1 x \tag{5}$$
$$M_2(x) = R_1 x - F_1(x-a) \tag{6}$$
$$M_3(x) = R_1 x - F_1(x-a) + F_2(x-b), \tag{7}$$

where each $M_i(x)$ equation is valid only for the range given in the accompanying free-body diagrams. If the coordinate system is taken from the right end, it may be convenient to use M_3 based on this alternate system: $M_3(x') = R_2 x'$.

Other information of interest in the solutions are to note that Young's modulus, E, is taken as 207,000 MPa and the area moment of inertia for a solid, round cross section is given by $I = \pi d^4/64$. Here, I_1 is based on $d_1 = 30$ mm, I_2 is based on $d_2 = 50$ mm and I_3 is based on $d_3 = 40$ mm.

THE SOLUTIONS

Superposition from Beam Table (By Hand Estimate)

As stated, the problem cannot be directly solved using superposition of beam table solutions because the cross section is stepped. However, the beam tables can be used to quickly determine an estimate for the solution—by hand calculation. The solution for a simply-supported beam with a single point force is given in all mechanics of materials and machine design texts.

We must choose a constant diameter (or alternatively, choose a constant moment of inertia) in making the calculations. Clearly, the 30 mm diameter is not representative of the entire shaft and we expect using it will lead to much larger deflections than the actual stepped shaft. The opposite thought comes to mind for the 50 mm diameter shaft; we expect that using 50 mm will lead to much smaller deflections than the actual stepped shaft. Perhaps, the 40 mm diameter is a good choice. For it, we expect a deflection that is smaller than the actual deflection for the leftmost part of the shaft. We also expect that the estimated deflection for the rightmost part of the shaft will be close to the actual deflection. Frankly, it is simple enough to try all three diameters and students should be encouraged to do so.

Successive Integration (Exact)

The successive integration of the moment equations requires three intervals. For each interval, there will be two constants of integration. Thus, there are six constants of integration to be found. For the simply-supported shaft, the boundary conditions $v(0) = 0$ and $v(L) = 0$ will provide two equations. Four additional equations are needed. They come from compatibility conditions at the intersections of the three intervals. At the left intersection, $x = a$, we must have $v(a)_{\text{left}} = v(a)_{\text{right}}$ and $v'(a)_{\text{left}} = v'(a)_{\text{right}}$. At the rightmost intersection, $x = b$, we must have $v(b)_{\text{left}} = v(b)_{\text{right}}$ and $v'(b)_{\text{left}} = v'(b)_{\text{right}}$.

From Figure 2, the moments can be integrated to yield slope and deflection equations. The integration results are summarized here. For $0 < x < a$,

$$M_1(x) = R_1 x \tag{8}$$

$$\int M_1(x)dx = EI_1\frac{dv_1}{dx} = R_1\frac{x^2}{2} + C_1 \tag{9}$$

$$\iint M_1(x)dx = EI_1 v_1(x) = R_1\frac{x^3}{6} + C_1 x + C_2. \tag{10}$$

For $a < x < b$,

$$M_2(x) = R_1 x - F_1(x-a) \tag{11}$$

$$\int M_2(x)dx = EI_2\frac{dv_1}{dx} = R_1\frac{x^2}{2} - F_1\frac{(x-a)^2}{2} + C_3 \tag{12}$$

$$\iint M_2(x)dx = EI_2 v_2(x) = R_1\frac{x^3}{6} - F_1\frac{(x-a)^3}{6} + C_3 x + C_4. \tag{13}$$

For $b < x < L$,

$$M_3(x) = R_1 x - F_1(x-a) + F_2(x-b) \tag{14}$$

$$\int M_3(x)dx = EI_3\frac{dv_1}{dx} = R_1\frac{x^2}{2} - F_1\frac{(x-a)^2}{2} + F_2\frac{(x-b)^2}{2} + C_5 \tag{15}$$

$$\iint M_3(x)dx = EI_3 v_3(x) = R_1\frac{x^3}{6} - F_1\frac{(x-a)^3}{6} + F_2\frac{(x-b)^3}{6} + C_5 x + C_6. \tag{16}$$

Here, the flexural rigidity EI and odd-subscripted constants C_1, C_3 and C_5 have the dimensions of $F \times L^2$ and the units of N $\times$ mm^2. Then even-subscripted constants C_2, C_4 and C_6 have the dimensions of $F \times L^3$ and units of N $\times$ mm^3.

Substituting $x = 0$ clearly yields $EI_1 v_1(0) = C_2$. But, $v_1(0) = 0$, so $C_2 = 0$. The other five conditions can be written in matrix form as

$$\begin{bmatrix} 0 & 0 & 0 & L & 1 \\ a & -\frac{a}{\alpha_2} & -\frac{1}{\alpha_2} & 0 & 0 \\ 1 & -\frac{1}{\alpha_2} & 0 & 0 & 0 \\ 0 & \frac{b}{\alpha_2} & \frac{1}{\alpha_2} & -\frac{b}{\alpha_3} & -\frac{1}{\alpha_3} \\ 0 & \frac{1}{\alpha_2} & 0 & -\frac{1}{\alpha_3} & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} -R_1\frac{L^3}{6} + F_1\frac{(L-a)^3}{6} - F_2\frac{(L-b)^3}{6} \\ \left(\frac{1}{\alpha_2} - 1\right) R_1 \frac{a^3}{6} \\ \left(\frac{1}{\alpha_2} - 1\right) R_1 \frac{a^2}{2} \\ \left(\frac{1}{\alpha_3} - \frac{1}{\alpha_2}\right)\left[R_1\frac{b^3}{6} - F_1\frac{(b-a)^3}{6}\right] \\ \left(\frac{1}{\alpha_3} - \frac{1}{\alpha_2}\right)\left[R_1\frac{b^2}{2} - F_1\frac{(b-a)^2}{2}\right] \end{Bmatrix}. \tag{17}$$

where $\alpha_2 = I_2/I_1$ and $\alpha_3 = I_3/I_1$. Note that Young's modulus E cancels out of the equations and does not appear in the above matrix. A brief MATLAB script for the successive integration method is given in Listing 1.

Castigliano's Second Theorem (Exact)

Castigliano's Second Theorem is a commonly used energy method. However, it is limited to finding the deflection at a single point. It requires a point force at the location of interest, but a fictitious point force can be used if the location does not have a point force. From Equations 5 through 7, we substitute the reaction equations 3-4 to yield:

$$M_1(x) = \left(\frac{7}{9}F_1 - \frac{1}{3}F_2\right)x \tag{18}$$

$$M_2(x) = \left(\frac{7}{9}F_1 - \frac{1}{3}F_2\right)x - F_1(x-a) \tag{19}$$

$$M_3(x) = \left(\frac{7}{9}F_1 - \frac{1}{3}F_2\right)x - F_1(x-a) + F_2(x-b). \tag{20}$$

Castigliano's Second Theorem requires partial derivatives of the moment equations with respect to the point force at the location of interest. If we use the method to determine the deflection at both $x = a$ (where F_1 is located) and $x = b$ (where F_2 is located), the following derivatives will be needed:

$$\frac{\partial M}{\partial F_1} = \begin{cases} \frac{7}{9}x & 0 < x < a \\ \frac{7}{9}x - (x-a) & a < x < b \\ \frac{7}{9}x - (x-a) & b < x < L \end{cases} \tag{21}$$

$$\frac{\partial M}{\partial F_2} = \begin{cases} -\frac{x}{3} & 0 < x < a \\ -\frac{x}{3} & a < x < b \\ -\frac{x}{3} + (x-b) & b < x < L. \end{cases} \tag{22}$$

The resulting equations for the deflections are

$$v(a) = \int_0^a \frac{M_1}{EI_1}\frac{\partial M_1}{\partial F_1}dx + \int_a^b \frac{M_2}{EI_2}\frac{\partial M_2}{\partial F_1}dx + \int_b^L \frac{M_3}{EI_3}\frac{\partial M_3}{\partial F_1}dx \tag{23}$$

and

$$v(b) = \int_0^a \frac{M_1}{EI_1}\frac{\partial M_1}{\partial F_2}dx + \int_a^b \frac{M_2}{EI_2}\frac{\partial M_2}{\partial F_2}dx + \int_b^L \frac{M_3}{EI_3}\frac{\partial M_3}{\partial F_2}dx. \tag{24}$$

The equations are implemented in a Maple script given in Listing 2. It is very important to to note to students that Castigliano's Second Theorem gives $v(b)$ as a negative value because the deflection is in the opposite direction of F_2. They must see that F_1 is downward and is twice as large as the upward F_2, so that the overall deflection at $x = b$ is still downward (negative).

The Maple script also includes the use of a fictitious force that can be varied in $a < x < b$ to determine deflections in that range. Thus, using a script makes the effort involved in Castigliano's Second Theorem more valuable.

Finite Element Method (Numerical Estimate)

The finite element method, described by Mueller [18] and many others, is a procedure that first decomposes a physical system into a set of simpler elements. Each element is governed by a simple system of algebraic equations. The elements are reassembled into an approximation of the original system, and the resulting system of equations is solved with matrix algebra. In the case of a structure subjected to loads within the linear elastic region of the material, each element is modeled as a linear elastic member with a governing equation of

$$\{f\}^{(e)} = [k]^{(e)} \{u\}^{(e)} \,, \tag{25}$$

where $\{f\}^{(e)}$ is the element load column vector, $[k]^{(e)}$ is the element stiffness matrix, and $\{u\}^{(e)}$ is the element displacement column vector. For the specific case of a stepped shaft with in-plane loads, each element can be represented similar to Figure 3, which shows element 1 connecting nodes 1 and 2.

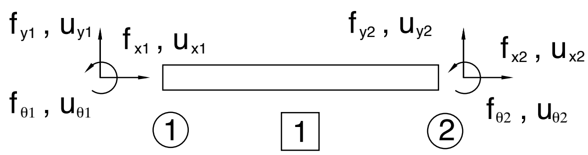

Figure 3: Schematic of beam element [18]

One method for constructing the element stiffness matrix is the direct method, which relates a unit displacement or rotation at a given node to the force required, assuming displacements and rotations at the other nodes in the element are fixed at zero. For example, from mechanics of materials, we know the deflection of a bar with an axial end load is $\delta = PL/AE$, and $P = \delta AE/L$. For a unit displacement of $\delta = 1$, $P = AE/L$. Thus, the stiffness matrix element relating a force at degree of freedom 1 to the corresponding deflection is AE/L. Using the same method on the other degrees of freedom for the element, the element stiffness matrix is defined as

$$[k]^{(e)} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^3} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^3} & \frac{4EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^3} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^3} & \frac{2EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{26}$$

The element stiffness matrices are assembled into a global stiffness matrix by summing up the contributions of each element stiffness matrix to each degree of freedom. For example, in a two-element model where element 1 contains nodes 1 and 2, and element 2 contains nodes 2 and 3, the global stiffness matrix $[K]$ would be

$$[K] = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & k_{15}^{(1)} & k_{16}^{(1)} & 0 & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} & k_{25}^{(1)} & k_{26}^{(1)} & 0 & 0 & 0 \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} & k_{35}^{(1)} & k_{36}^{(1)} & 0 & 0 & 0 \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} + k_{11}^{(2)} & k_{45}^{(1)} + k_{12}^{(2)} & k_{46}^{(1)} + k_{13}^{(2)} & k_{14}^{(2)} & k_{15}^{(2)} & k_{16}^{(2)} \\ k_{51}^{(1)} & k_{52}^{(1)} & k_{53}^{(1)} & k_{54}^{(1)} + k_{21}^{(2)} & k_{55}^{(1)} + k_{22}^{(2)} & k_{56}^{(1)} + k_{23}^{(2)} & k_{24}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\ k_{61}^{(1)} & k_{62}^{(1)} & k_{63}^{(1)} & k_{64}^{(1)} + k_{31}^{(2)} & k_{65}^{(1)} + k_{32}^{(2)} & k_{66}^{(1)} + k_{33}^{(2)} & k_{34}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} \\ 0 & 0 & 0 & k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} & k_{45}^{(2)} & k_{46}^{(2)} \\ 0 & 0 & 0 & k_{51}^{(2)} & k_{52}^{(2)} & k_{53}^{(2)} & k_{54}^{(2)} & k_{55}^{(2)} & k_{56}^{(2)} \\ 0 & 0 & 0 & k_{61}^{(2)} & k_{62}^{(2)} & k_{63}^{(2)} & k_{64}^{(2)} & k_{65}^{(2)} & k_{66}^{(2)} \end{bmatrix} , \tag{27}$$

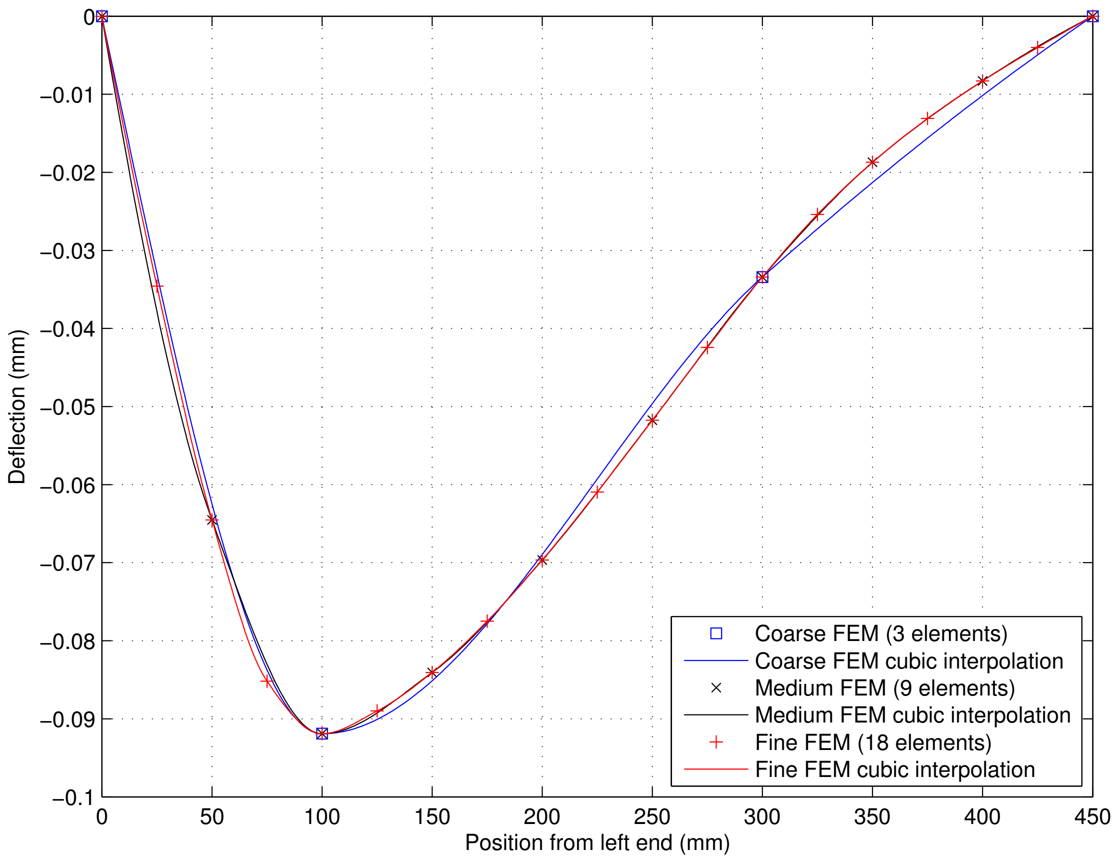

Figure 4: Influence of element length on shaft deflection

where $k_{ij}^{(m)}$ represents the ith row and jth column of the stiffness matrix for element m.

Once the global stiffness matrix is assembled, the global force and displacement vectors ($\{F\}$ and $\{U\}$) are constructed. Each vector is a list of known and unknown forces and displacements sorted by degree of freedom. Since the first goal of the finite element method is to solve for the unknown displacements, all items in the stiffness, force, and displacement matrices corresponding to known displacements must be eliminated. For example, if node 1's y displacement (corresponding to degree of freedom 2) is fixed, then row 2 of the global force and displacement vectors must be eliminated, along with row 2 and column 2 of the global stiffness matrix.

Now that the global matrices and vectors contain only unknown displacements, the system of equations can be solved for $\{U\}$. Any method for solving a system of equations $\{F\} = [K]\{U\}$ for $\{U\}$ is acceptable, but Gaussian elimination or LU decomposition are popular methods. A MATLAB script demonstrating this procedure, adapted from [18], is given in Listing 3. The mesh provided in the script uses an element length of 50 mm for a total of nine elements. This discretization is fine enough to provide an excellent match to the successive integration and Castigliano's solutions. The script should be reused with fewer elements (perhaps three elements of unequal length) and with a larger number of elements (perhaps 18 elements of a uniform length of 25 mm) to demonstrate the convergence process, as shown in Figure 4.

Experience, Results Summary and Conclusions

The first author's experience with the example problem in a machine design course has been very successful. Students appreciate having a second (or third or even fourth) chance to successfully solve a problem. The example has been given to the class in three ways. First, the example has been used as an in-class case study spanning parts of lectures over a three-week period of studying design for stiffness principles. The total time expended for the example given here is approximately three hours (approximately one-third of the total time allocated for design for stiffness in

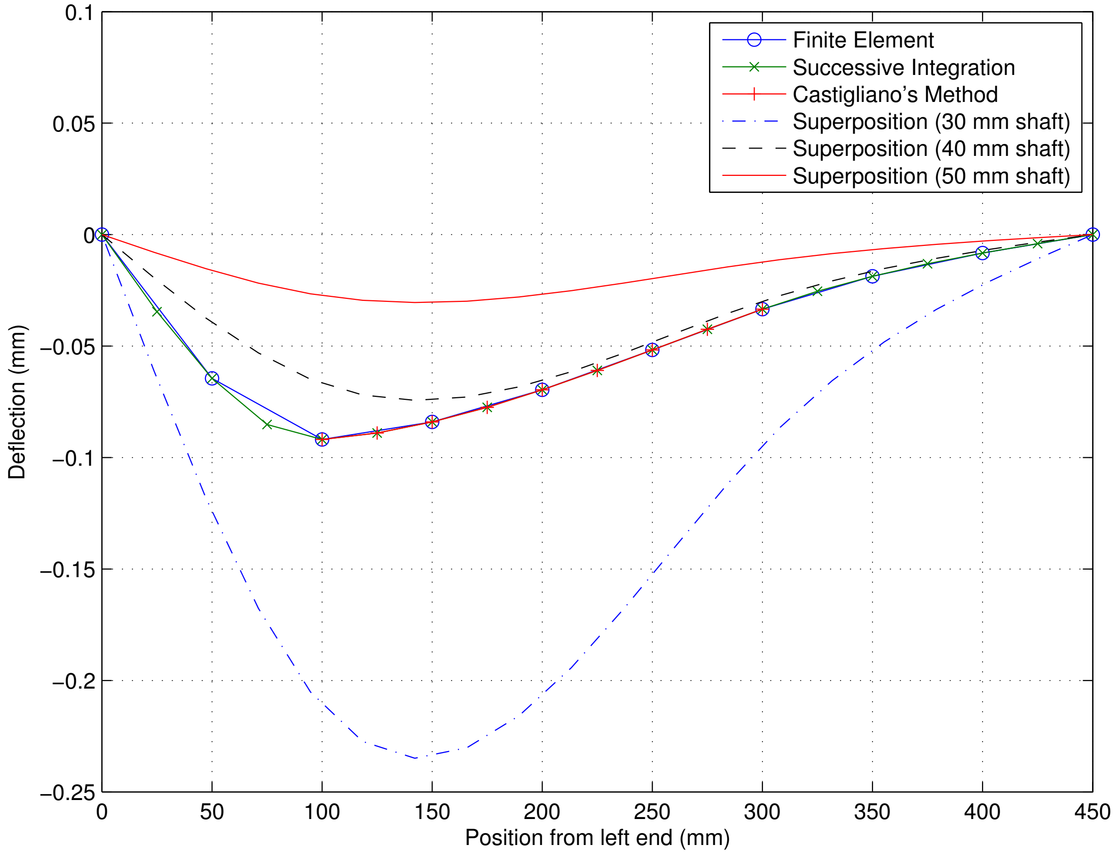

Figure 5: Comparison of methods for calculating shaft deflection

the course syllabus). Second, the example (and other similar ones) has been used in successive homework assignments and quizzes. With this approach, students worked through three homework assignments in a series, having feedback (grades) before tackling the next assignment. The third way was a review problem set/example given at the end of the term in preparation for the final exam—no class time was expended. The anecdotal responses from students was good for all three approaches. However, no quantitative measurements of student improvement have been made.

A complete comparison of results is given in Figure 5. Straight lines are used between calculated points in the figure for simplicity. However, it should be pointed out, that for the finite element results, that the solution between adjacent nodes is actually a cubic. The estimates based on the beam table superposition verify that the actual solution is bounded between the 30 mm and the 50 mm diameter estimates. The 40 mm diameter estimate is quite accurate for this problem. Students should be warned that estimates are not always this close to the exact solution.

The first author has also prepared a solution using numerical integration (trapezoid rule). This solution is implemented in a spreadsheet (Excel) and closely follows the method laid out in the fifth edition of Shigley [10]. The numerical integration method may be a good alternative to using the finite element method for those less familiar to finite elements or to those concerned that the finite element method will be considered as a black-box to the students.

No emphasis was placed on the slope of the deflected shaft. The slope will be important in machine design calculations involving bearings. Some additional effort can be spent on this detail if desired.

In conclusion, the stepped shaft problem is sufficiently complicated that the various solution methods presented here are actually useful. Hopefully, the problem is not overly complicated. A similar effort for a statically indeterminate beam or shaft would also be useful.

REFERENCES

[1] Gere, James M. and Stephen P. Timoshenko. *Mechanics of Materials*. PWS, Boston, 1997.

[2] Popov, Egor P. *Mechanics of Materials*. Prentice Hall, Upper Saddle River, New Jersey, 1976.

[3] Beer, Ferdinand P., Jr. E. Russell Johnston, John T. DeWolf, and David F. Mazurek. *Mechanics of Materials*. McGraw-Hill, New York, 2011.

[4] Higdon, Archie, Edward H. Ohlsen, and William B. Stiles. *Mechanics of Materials*. Wiley, Hoboken, New Jersey, 1960.

[5] Riley, William F., Leroy D. Sturges, and Don H. Morris. *Mechanics of Materials*. Wiley, Hoboken, New Jersey, 2006.

[6] Philpot, Timothy A. *Mechanics of Materials: An Integrated Learning System*. Wiley, Hoboken, New Jersey, 2012.

[7] Vable, Madhukar. *Mechanics of Materials*. Oxford University Press, New York, 2002.

[8] Allen, David H. *Introduction to the Mechanics of Deformable Solids: Bars and Beams*. Springer, Berlin, 2012.

[9] Hibbeler, Russell C. *Mechanics of Materials*. Prentice Hall, Upper Saddle River, New Jersey, 2010.

[10] Shigley, Joseph Edward and Charles Mischke. *Mechanical Engineering Design*. McGraw-Hill, New York, 1989.

[11] Phelan, Richard M. *Fundamentals of Mechanical Design*. McGraw-Hill, New York, 1970.

[12] Spotts, Merhyle F., Terry E. Shoup, and Lee E. Hornberger. *Design of Machine Elements*. Prentice Hall, Upper Saddle River, New Jersey, 2003.

[13] Johnson, Ray C. *Optimum Design of Mechanical Elements*. Wiley, New York, 1980.

[14] Norton, Robert L. *Machine Design*. Prentice Hall, Upper Saddle River, New Jersey, 2010.

[15] Ugural, Ansel C. *Mechanical Design: An Integrated Approach*. McGraw-Hill, New York, 2003.

[16] Collins, Jack A., Henry R. Busby, and George H. Staab. *Mechanical Design of Machine Elements and Machines*. Wiley, Hoboken, New Jersey, 2009.

[17] Juvinall, Robert C. and Kurt M. Marshek. *Fundamentals of Machine Component Design*. Wiley, Hoboken, New Jersey, 2011.

[18] Donald W. Mueller, Jr. "Introducing the Finite Element Method to Mechanical Engineering Students Using MATLAB." In "Proceedings of the 2003 American Society for Engineering Education Annual Conference & Exposition," American Society for Engineering Education, 2003.

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SUCESSIVE INTEGRATION CODE

Listing 1: MATLAB Script for Successive Integration

```
% Successive integration method for stepped shaft

% Define the symbolic problem
syms x1 x2 x3 E L a b d1 d2 d3 F1 F2 c1 c2 c3 c4 c5 c6
R1 = (L-a)/L*F1-(L-b)/L*F2;
I1 = pi*d1^4/64;
I2 = pi*d2^4/64;
I3 = pi*d3^4/64;
alpha2 = I2/I1;
alpha3 = I3/I1;
coeff = [ ...
    0  0          0          L          1;
    a -a/alpha2 -1/alpha2  0          0;
    1 -1/alpha2  0          0          0;
    0  b/alpha2  1/alpha2 -b/alpha3 -1/alpha3;
    0  1/alpha2  0         -1/alpha3  0 ...
    ];
rhs = [-R1*L^3/6+F1*(L-a)^3/6-F2*(L-b)^3/6;
    (1/alpha2-1)*R1*a^3/6;
    (1/alpha2-1)*R1*a^2/2;
    (1/alpha3-1/alpha2)*(R1*b^3/6-F1*(b-a)^3/6);
    (1/alpha3-1/alpha2)*(R1*b^2/2-F1*(b-a)^2/2)];
C = coeff\rhs; % elements of C1, C3, C4, C5, C6
C = [C(1); 0; C(2:5)]; % elements of C1 ... C6
v1 = (R1*(x1^3/6)+C(1)*x1+C(2))/(E*I1);
v2 = (R1*(x2^3/6)-F1*(x2-a)^3/6+C(3)*x2+C(4))/(E*I2);
v3 = (R1*(x3^3/6)-F1*(x3-a)^3/6+...
    F2*(x3-b)^3/6+C(5)*x3+C(6))/(E*I3);

% Parameters for this particular stepped shaft
L = 450; a = 100; b = 300;
d1 = 30; d2 = 50; d3 = 40;
F1 = 4000; F2 = 2000; E = 207e3;
x1 = 0:25:a; x2 = a:25:b; x3 = b:25:L;
% Evaluated solution
v1evaluated = subs(v1);
v2evaluated = subs(v2);
v3evaluated = subs(v3);
plot(x1,v1evaluated,'-o',x2,v2evaluated,'-s',...
    x3,v3evaluated,'-d')
xlabel('Position along shaft (mm)');
ylabel('Deflection (mm)');
grid on;
table = [ x1 x2 x3; v1evaluated v2evaluated v3evaluated ]'
```

Castigliano's Method Code

Listing 2: Maple Script for Castigliano's Method

```
restart;
# Set up equilibrium equations
Fy:=R1+R2-F1-F2-P:
M0:=-F1*a-F2*b-P*c+R2*L:
# Solve for reactions
R2:=solve(M0=0,R2);
R1:=solve(Fy=0,R1);
# Define moment diagram
M1:=R1*x:
M2:=M1-F1*(x-a):
M3:=M2-P*(x-c):
M4:=M3-F2*(x-b):
# Take partials of moments with respect to P
dM1dP:=diff(M1,P):
dM2dP:=diff(M2,P):
dM3dP:=diff(M3,P):
dM4dP:=diff(M4,P):
# Define deflection at P in terms of partial of strain energy with
# respect to P
vP:=int(M1/(E*I1)*dM1dP,x=0..a)+int(M2/(E*I2)*dM2dP,x=a..c)+ \
    int(M3/(E*I2)*dM3dP,x=c..b)+int(M4/(E*I3)*dM4dP,x=b..L):
# Define shaft geometry, material, and loading
d1:=30: d2:=50: d3:=40: a:=100: b:=300: L:=450:
I1:=evalf(Pi)*d1^4/64: I2:=evalf(Pi)*d2^4/64: I3:=evalf(Pi)*d3^4/64:
E:=207e3: F1:=4000: F2:=-2000: P:=0:
# Evaluate deflection for given shaft and loading
for c from a to b by 25 do
    v:=evalf(vP);
    printf("x=%f, v=%f\n", c, v);
end do:
```

FINITE ELEMENT METHOD CODE

Listing 3: MATLAB Script for Finite Element Method

```
clear all;
close all;
% All units in terms of N and mm (including MPa)
% Shaft geometry
d1 = 30;  d2 = 50;  d3 = 40;  % shaft diameters
L1 = 100; L2 = 200; L3 = 150; % shaft lengths
% Element geometry
Le = 50; % element length, must be divisor of L1, L2, L3
nElements = (L1+L2+L3)/Le; % number of elements
L = Le*ones(1,nElements); % vector of element lengths
d = [d1*ones(1,L1/Le) ...
     d2*ones(1,L2/Le) ...
     d3*ones(1,L3/Le)]; % vector of element diameters
A = pi*d.^2/4; % vector of element cross-sectional area
I = pi*d.^4./64; % vector of element moment of inertia
nNodes = nElements + 1; % number of nodes
nDof = 3*nNodes; % degrees of freedom
% Boundary conditions
F1 = -4e3; F2 = 2e3; % point loads
pinnode = 1; % node at left end of shaft
rollernode = nNodes; % node at right end of shaft
F1node = (L1/Le)+1; % node between first and second segments
F2node = ((L1+L2)/Le)+1; % node between second and third segments
% Material properties
E = 207e3*ones(1,nElements); % vector of element modulus of elasticity
% Step 1: Construct element stiffness matrix
k = zeros(6,6,nElements);
for n = 1:nElements
    k11 = A(n)*E(n)/L(n); k22 = 12*E(n)*I(n)/L(n)^3;
    k23 = 6*E(n)*I(n)/L(n)^2; k33 = 4*E(n)*I(n)/L(n);
    k36 = 2*E(n)*I(n)/L(n);
    k(:,:,n) = [ ...
         k11    0    0 -k11    0    0;
           0  k22  k23    0 -k22  k23;
           0  k23  k33    0 -k23  k36;
        -k11    0    0  k11    0    0;
           0 -k22 -k23    0  k22 -k23;
           0  k23  k36    0 -k23  k33];
end
% Step 2: Combine element stiffness matrices to form global stiffness
% matrix
shift = 0;
Ke = zeros(nDof,nDof,nElements);
for n = 1:nElements
    for i = 1:6
        for j = 1:6
            Ke(i+shift,j+shift,n) = k(i,j,n);
        end
    end
    shift = shift + 3;
end
K = sum(Ke,3);
```

```
F = zeros(nDof,1);
F(3*(F1node-1)+2) = F1;
F(3*(F2node-1)+2) = F2;
% Step 3: Reduce global stiffness and force matrices with constraints --
% remove columns from right to left, and rows from bottom to top.
Kr = K; Fr = F;
Kr(:,3*(rollernode-1)+2) = []; Kr(3*(rollernode-1)+2,:) = [];
Kr(:,3*(pinnode-1)+2) = []; Kr(3*(pinnode-1)+2,:) = [];
Kr(:,3*(pinnode-1)+1) = []; Kr(3*(pinnode-1)+1,:) = [];
Fr(3*(rollernode-1)+2) = [];
Fr(3*(pinnode-1)+2) = [];
Fr(3*(pinnode-1)+1) = [];
% Step 4: Solve for unknown displacements
Ur = Kr\Fr;
% Step 5: Solve for forces -- first augment Ur with constrained
% displacements
U = [ Ur(1:3*(pinnode-1)); 0 ; 0; ...
    Ur(3*(pinnode-1)+1:3*(rollernode-1)-1); ...
    0; Ur(3*(rollernode-1):end) ];
F = K*U;
x = Le*(0:nNodes-1);
Uy = U(2:3:end);
plot(x,Uy,'b+-'); grid on; title('Simply supported stepped shaft');
xlabel('Position along shaft (mm)'); ylabel('Deflection (mm)');
```