An Interactive Tool for Visually Presenting Conic Sections to STEM Students

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Abstract – Students in Science, Technology, Engineering, and Mathematics (STEM) fields are particularly at-risk of changing majors when they are enrolled in introductory mathematics courses. The use of visual lessons has been shown to keep students' interest. We have developed software that provides a simple and practical visual tool for use by professors and students alike in their exploration of the mathematics behind conic sections. This paper describes a new method that finds a one dimensional parametric solution for the implicit quadratic function, $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, that is both accurate and fast. This permits development of smoothly-animated interactive software that demonstrates the continuity of solutions among parabolas, ellipses, and hyperbolas by allowing a user to click and drag five control points while the conic section that connects them is shown in real time. Demonstrations enhance traditional teaching methods by showing the connection among the different conic sections in a visually-compelling way.

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BACKGROUND

There remains a demand and need for all types of engineering graduates in both the public and private sector job market [Goudreau, 6]. However, low retention rates hurt the efforts of engineering schools to increase graduation numbers and fulfill this need. The question remains whether retention can be increased to help meet this demand while still graduating competent engineers.

Engineering students normally start in a pre-calculus or calculus course their first semester of college. It has been shown that these courses bore many students and cause some to become discouraged and discontinue on an engineering career path [Lowery, 10; Hensel, 8]. In the past some programs have viewed this as a natural attrition process that helps them retain only the most motivated and prepared students. However, this has led to retention rates for many engineering schools between 33 and 50 percent [Wood, 14; Buechler, 1]. These statistics indicate that assistance in math courses could help students during their first few years of engineering school and possibly even increase retention.

There are many underlying problems that if addressed might help capable students succeed as engineering majors. In many cases the students who enter college are not prepared for the rigors required of engineering majors [Hanushek, 7]. This lack of preparation translates into negative effects in math, science, and future engineering courses [Wood, 14; Radzi, 11]. Students' lack of success and interest in introductory math may have deep seated biases formed from years in a secondary education system that did not prepare or challenge them. The goal should be to motivate these students to study harder, learn the deficiencies they may have in math, and realize they can still succeed.

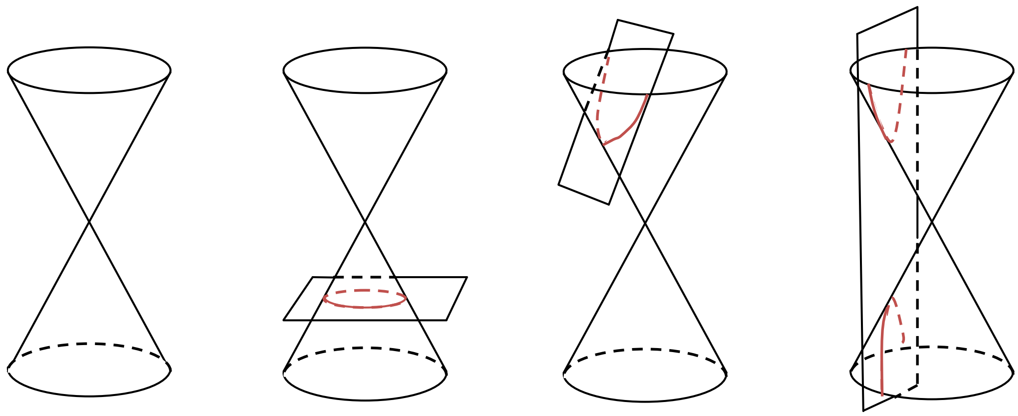
When educating these students in mathematics the question becomes: what techniques can professors use to help connect students' worlds with what they learn? Students who understand the math they are using are less likely to drop out of engineering. Any technique that accomplishes this early in a student's career before he or she considers quitting can help in retention.

In addition to keeping students interested in engineering, it is important to help students grasp the mathematical concepts so they can be used in the future. The outcome should be to retain students who are learning the key fundamental math concepts not just retaining more unprepared students who do not understand mathematics. Students who master mathematics concepts are better suited to understand future engineering courses and apply them to real world situations. The first step in doing this is to present math in new ways that are visual and interactive [Diehl, 2]. With the universal availability of personal computing, new methods should be used that make learning more efficient and effective.

Studies have attempted to use summer programs before college begins or rehabilitation programs to help connect mathematic concepts to engineering problems [Hensel, 8; Gleason, 4]. While these programs have great value they also require substantial costs, time, and may delay student's graduation. For many students simply teaching with different technologies may help keep them motivated in their classes and help them learn more effectively [Buechler, 1; Li 9]. Visual methods aid in the instruction of abstract concepts commonly found in mathematics [Ferguson, 3]. The use of plotting software can help demonstrate and model how math concepts are used when applied to various engineering topics [Gossens, 5; Smerage, 12]. The objective is to devise these new tools that augment the learning process in a simple and cost effective manner using technology that is already available.

CONIC SECTIONS

One mathematics concept that is commonly taught with many engineering applications is the conic section [Stewart, 13]. Conic sections are represented by stacking two cones connected at their apex (Figure 1 - A). Planes are drawn through the cones at various angles creating ellipses, parabolas, and hyperbolas at their intersection (Figure 1 – B, C, and D, respectively). It can then be shown that the conic sections describe many concepts in various areas of engineering from growth rates and magnetic fields to projectile paths and transportation curves.



A - Cones B - Ellipse C - Parabolas D - Hyperbolas

Figure 1 – Conic Sections Created by Slicing Cones

Visualizing the geometry of each unique shape can be accomplished through two dimensional space as shown in Figure 1. Many times students are taught to plot these individual slices of the conic section with graphing calculators or computerized plotting tools. This has become a common method that helps student visualize the various mathematical equations but it does not show the relation among all conic sections.

A tool is needed to efficiently plot various conic shapes and see the connection between each plot. The equation to describe all of these shapes is the generalized quadratic formula:

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

The solution transitions from a circle to an ellipse to a parabola and then a hyperbola as shown above. A horizontal plane (zero degree slope) that intersects a cone creates a circle. As the angle increases relative to the horizontal, the conic shape transitions into an ellipse. It eventually reaches a break point when it becomes a parabola (when the plane is parallel to the cones) and then afterwards, the conic shape becomes a hyperbola. These transitions are a natural progression that are visually interconnected and mathematically related. Further, a number of degenerate solutions naturally arise. A plane of the same inclination as the cones that passes through the origin will yield a single line, and tilting the plane steeper causes a loci of intersection points describing a pair of intersecting straight lines.

To the author's knowledge no simple tools exist that can demonstrate the interactions between the different conic sections made by cutting sections through cones. Currently available algorithms either compute these intersections using symbolic solutions which are computationally intensive, or approximate the solutions by plotting the contour line where the isosurface z = 0 intersects with the function z = f(x,y) = $Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, which is numerically ill-conditioned when the gradient of f(x,y) is nearly zero.

SOFTWARE

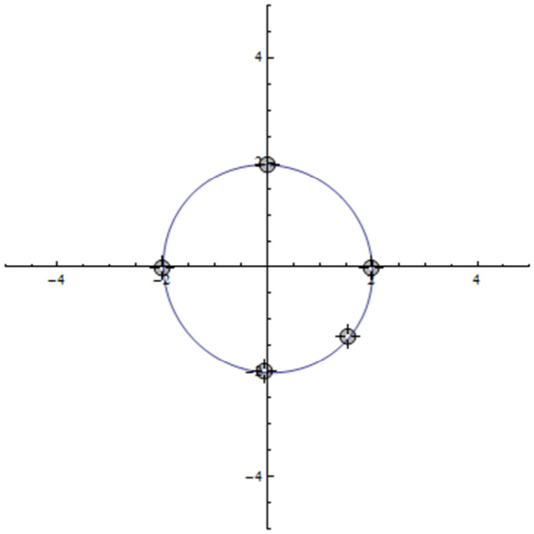
A new math tool has been developed based on an implicit quadratic function. The generalized quadratic function is solved with a linear algebra solution. The method first analyzes the coefficients of the generalized quadratic formula to detect degenerate behavior: a single line, crossed lines, or the situation where two or more points coincide. The single and crossed lines are plotted parametrically; the case of coincidental points returns an empty graph as the uniqueness of the conic is lost. Again, it is important here that the program doesn't "crash" or return an "error" message, implying the situation is "wrong" or "hard" to deal with.

Non-degenerate conics are easy to plot when their axes of symmetry are parallel to the *x* and *y* axes, i.e., when the "*Bxy*" term in the generalized quadratic formula is 0. When this term is not zero, the resulting conic can be viewed as a rotated-and-shifted version of a conic centered at the origin whose axes of symmetry are the coordinate axes. The angle of rotation and vector of shift are again obtainable from the coefficients. The given non-degenerate conic is thus plotted according to the following algorithm:

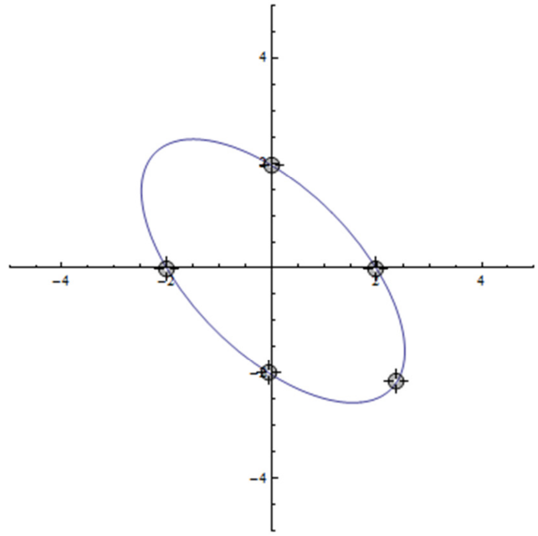
1. Determine the angle of rotation and shift vector.
2. Create a new, equivalent conic which is centered at the origin with axes of symmetric along the coordinate axes.
3. Plot the new conic parametrically. (In each of the following, a and b represent scalars.)
   a. Ellipses are plotted as ($a \bullet \cos\theta$, $b \bullet \sin\theta$).
   b. Hyperbolas are plotted as ($a \bullet \tan\theta$, $b \bullet \sec\theta$).
   c. Parabolas are plotted as (t, $a \bullet t^2$).
4. Rotate and shift the plot back to its original location and orientation.

The above techniques rely on simple, algebra-based computations based on the coefficients of the generalized quadratic formula and the use of standard trigonometric functions. The program provides a quick and accurate solution regardless of the type of conic section produced. Only linear algebra techniques are used to solve the conic section; there is no root-finding or function minimization, so the results are computed and displayed almost instantaneously on a laptop computer.

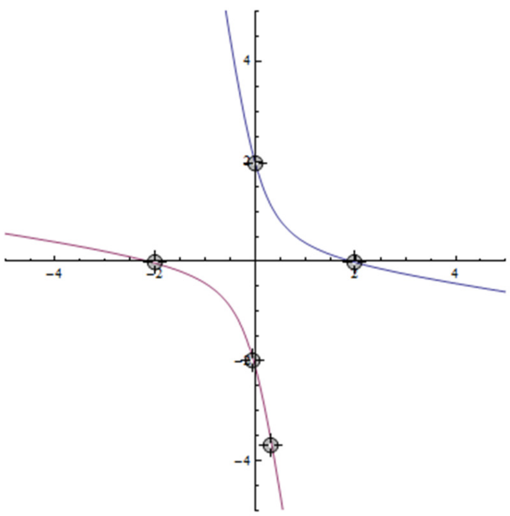
The points may then be moved one at a time to demonstrate how a conic section maneuvers from one type to another (Figure 2). The smoothing function automatically makes the adjustments in real time. For example in Figure 2 - A the five points are initially plotted as a circle. If the point in the lower right quadrant (1.7, -1.7) is dragged downward and to the right the shape instantaneously transitions to an ellipse (Figure 2 – B). By dragging this point downward and toward the vertical axis, the plot transitions and becomes a hyperbola (Figure 2 – C). The parabola is created by returning to the original 5 points in Figure 2 – A and moving the point on the vertical axis at (0,2) to the lower left hand quadrant. When the point is at (-1.7, -1.7) the parabola is formed instantaneously.



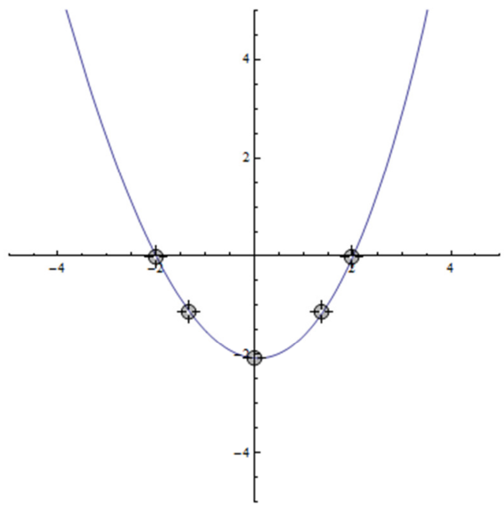
A – Circle



B - Ellipse



C – Hyperbola



D - Parabola

Figure 2 – Plotted Shapes using the Math Tool

One of the benefits of this program is its easy accessibility. A professor or student with a laptop computer can easily obtain and use this program. It does not require any specialized training and fits in with courses that already use math software. There are no specialized software costs because it runs on programs that most professors and many students already have access to on their personal computers.

The program is written in Mathematica code in an open source format. As a student progresses, or if a professor desires, the code could be changed or adjusted to perform additional functions. In the long term the program could be developed to solve many other functions.

PEDAGOGICAL USE

The program is designed so it may be used as a demonstration tool in the classroom. A professor can introduce the conic concepts in a math class visually instead of simply plotting equations or showing static section cuts on a Powerpoint or chalkboard. This program demonstrates in real time the various shapes defined by the conic sections. A professor could continue to use this tool from the introduction of conic concepts through the presentation of the detailed solutions for each section. The mathematics behind the sections could simultaneously be displayed and related to the visuals during the learning process.

The program may also be used independently by students during their personal study time. Its ease of use allows students to quickly replicate demonstrations done during class. It also allows them to check homework or example problems in a graphical manner. The results are visually compelling and exciting because of the program's ease of use and dynamic display.

The program has recently been developed and is in the implementation stage. The goal is to provide this program to math professors and students and measure how effective it is in various settings. Students' results will be monitored along with their perceptions of the program during the class.

CONCLUSIONS

Recruitment and retention of engineering students starts with their first mathematics courses in college. If students cannot visualize the concepts and see their relevant use, then many times they perform poorly or lose interest in engineering majors. A host of ideas have been presented to try to revise curriculum and student preparation. New concepts that are simple and cost effective are needed that connect mathematical concepts with visual examples.

A program that plots conic sections has been developed that is efficient, inexpensive, and easy to use. This program runs on a simple platform available to most students and can be used by both an instructor and student. Visual representations of all kinds of conic sections are quickly plotted within the program. Users may then instantaneously plot various types of solutions and see the interconnection of the results. The goal is to implement this in mathematics, particularly in calculus classes, when students are learning these fundamental concepts. As they see how conic sections are manipulated in real time, the students can better visualize and connect the theories behind them to real world concepts.

This program is a step toward implementing new visual teaching techniques into math classrooms that are efficient and easy to use by engineering students while still being cost effective. The eventual implementation of this tool and similar ones should help students in their math courses and increase retention of engineering students during the early critical years of college.

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