A First-Order Approximations Module for a First-Year Engineering Course

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Abstract – First-order approximations are important mathematical tools which find application across science and the engineering disciplines. They are the heart of the common technique of linearizing nonlinear systems around an operating point. Applications range from the simple pendulum to analog electronic circuits to control systems. In addition, as is well-known, the mathematics skills of aspiring engineers are critical to future success, and, as a result, many first-year engineering programs place considerable emphasis on developing students' mathematics skills, supplementing or even replacing traditional mathematics instruction. This paper describes a first-semester engineering course module on first-order approximations. The engineering course assumes only a high school precalculus background and does not employ calculus. The course supplements but does not replace traditional mathematics instruction and is typically taken concurrently with the first calculus course. While first-order approximations are related to Taylor series expansions, a topic typically covered in a second or third term calculus course, first-order approximations are themselves simple, algebraic formulas. This module explores the first-order approximations of a number of functions which occur frequently in engineering, such as $(1+x)^n$, e^x, sin x, and tan x. They are explored computationally rather than theoretically. They are applied to specific problems to compute approximate solutions. Applications include estimating fuel mileage, square roots, and exponential growth and decay. This exercises students' precalculus math skills, builds their skills at applying mathematics, strengthens their understanding of nonlinear functions, provides tools for making "back of the envelope calculations," and prepares them to see these concepts in greater depth later. This module was first implemented in the 2010 fall term, so the paper's results are still preliminary. The desired outcomes are: 1) enhanced retention and 2) improved performance in subsequent technical courses, most immediately in foundational mathematics and physics courses. Retention in the engineering major and course grades in Calculus I and Physics I will be used for assessment.

Keywords: Freshman programs, mathematics, first-order approximations.

OVERVIEW

Because the mathematics skills of aspiring engineers are critical to future success, many first-year engineering programs place considerable emphasis on developing students' mathematics skills, supplementing or, as in the Wright State University Model, even radically restructuring traditional mathematics instruction [1]-[4]. This paper describes a module on first-order approximations which was developed as part of a one-semester, 2-credit hour "Intro to Engineering" course for first-semester freshman engineering majors.

The course, EGR 101 "Introduction to Engineering Design and Analysis," introduces students to the engineering profession, engineering design, problem-solving, and other concepts and skills important for their success in their college and professional careers. Selected mathematics topics are included in "other concepts and skills." The course assumes only a high school precalculus background, does not employ calculus, and operates alongside traditional calculus-sequence courses. Precalculus or first-semester calculus is typically taken concurrently with EGR 101, along with first-semester chemistry, whereas Physics I is typically taken the following semester. Over the years, EGR 101 has included such math-oriented topics as number notation, graphing, sources of error, significant figures, uncertainty, and least-squares linear regression. Both the problem-solving and mathematical topics serve two purposes: 1) students gain familiarity with tools useful in engineering; and 2) students apply their skills in

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algebra, trigonometry, exponentials, and logarithms to engineering-related problems. The first-order approximations module was developed to serve these two purposes as well.

First-order approximations are themselves important mathematical tools which find application across science and the engineering disciplines. A first-order approximation is key to the classic analysis of the simple pendulum. Likewise, first-order approximations are the heart of the common technique of linearizing nonlinear systems around an operating point as in the analysis and design of electronic circuits and control systems. While first-order approximations are related to Taylor series expansions, a topic typically covered in a second or third term calculus course, first-order approximations are simple, algebraic formulas. The module explores the first-order approximations of a number of functions which occur frequently in engineering, such $(1+x)^n$, e^x, sin x, and tan x. They are explored computationally rather than theoretically. They are applied to specific problems to compute approximate solutions. Applications include fuel mileage, square roots, and exponential growth and decay. This exercises students' math skills, builds their skills at applying mathematics, strengthens their understanding of nonlinear functions, provides tools for making "back of the envelope calculations," and prepares them to see these concepts in greater depth later. This module was implemented for the first time in the 2010 fall term and repeated in the 2011 fall term (EGR 101 is offered only in the fall term), so this paper's results are preliminary. The desired outcomes are: 1) enhanced retention and 2) improved performance in subsequent technical courses, most immediately in foundational mathematics and physics courses. First-year to second-year retention and course grades in Calculus I and Physics I will be used for assessment.

MODULE DESCRIPTION

The module consists of two 50-minute lectures, accompanied by a homework assignment. The specific functions included in the module are shown in Tables A-I and A-II in the Student Handout in Appendix A of this paper. In class, the material is first motivated by a numerical example in which a first-order approximation simplifies the calculation of travel time given the distance and average speed. This example, along with the other examples worked in class, is included in the Student Handout in Appendix A. For completeness, after the initial example, a brief qualitative description is made of power series (without proof). Power series and first-order approximations for the other common functions are then introduced.

As shown in Appendix A, examples worked in-class apply first-order approximations to the calculation of travel time, fuel economy, square roots, exponential growth, pH, and the height of football goalposts. These examples apply first-order approximations for $(1+x)^n$, e^x, sin x, and tan x. They require knowledge of basic algebra, exponentials, logarithms, and trigonometric functions, and they require some creativity to manipulate the basic problem into a form suitable to the application of a first-order approximation.

Initially in lecture, it is simply asserted that first-order approximations are valid for "small x," which is broadly interpreted to mean that the magnitude of x is much less than one: "$|x| << 1$." That is, the first-order approximations are implicitly based on Maclaurin expansions of the functions – linearized around 0. After a number of examples have been worked, the question "How small is 'small'?" is addressed in two ways. First is a purely numerical approach: plot a specific function, its first-order approximation, and the percent error versus x. An example of this is shown in "Figure 1" in the Student Handout in Appendix A. This gives insight into the behavior of the particular function and its approximation, and the error for a given "small-ness" of x is clear.

The second approach introduces a common engineering rule of thumb which interprets "much less than" to mean "less than by at least a factor of 10." That is "$|x| << 1$" is interpreted to mean "$|x| < 10^{-1}$." This is justified by referring back to the general power series formula to argue that if x is "small" – on the order of 10^{-1} – then x^2, the next term in the series, is "very small" – on the order of 10^{-2}. This is borne out by plots like "Figure 1" in Appendix A. Therefore, without detailed knowledge of a function, a good rule of thumb is that first-order approximations yield reasonably accurate results when $|x|$ is 0.1 or smaller.

The homework assignment is included in Appendix B. Students create plots of two functions, their first-order approximations, and the percent errors. Students also apply first-order approximations to estimate: the side of a square given its area; the side of a cube given its volume; the capacitor voltage as a function of time in an RC circuit; the value of π; and the voltage across a non-linear two-terminal device (a semiconductor diode) as the current is varied around an operating point. The problems are designed to be self-checking in that the student knows or is required to calculate exact values along with approximate values.

RESULTS

Because this module was first implemented in the fall 2010 term and has run only two times, the results must be taken as preliminary. The module's effect on student retention and performance will need to be monitored over an extended period. Moreover, it is expected that this module's effect will be modest, in light of the fact that it constitutes one week of material in a fourteen week course.

The students' apprehension of the first-order approximations material was only fair at best, as indicated by scores on the homework assignment. In 2010, the average score on the homework assignment was 78%, a C on a 10-point grading scale. This was actually slightly above the overall homework average of 75%. In 2011, the average score on the first-order approximations homework assignment was 59%, compared to 68% overall on homework. Several students did not turn in the assignment in 2011, which brought this average down. Overall, the first-order approximations homework assignment average was 66%, a D, compared to an overall homework average of 70%.

Table I summarizes student retention and performance for the two years prior to implementation and the year and a half since implementation. Data from only two years' before implementation is shown, because, while the general format of EGR 101 has been unchanged since AY 2005-2006, significant retention-enhancement initiatives that were implemented in succeeding years had stabilized by AY 2008-2009. Table I also includes relevant auxiliary retention and student profile data for reference. The significant retention-enhancement initiatives implemented prior to AY 2008-2009 were: 1) introducing a Calculus I placement test and 2) postponing Physics I until after Chemistry I and Calculus I (previously, Calculus I and Physics I were taken concurrently the first semester). The only other retention-enhancement initiatives implemented in the time-frame shown in Table I were: 1) creating an "Engineering Student Handbook" (AY 2009-2010) and 2) holding a "New Engineering Student Orientation" (AY 2010-2011).

The key first-year to second-year retention rate was 86% for the first year the first-order approximations module was introduced, 21 percentage points above the cumulative rate prior. Data is not yet available for the second year, but the intermediate first-to-second semester retention rate (74% for 2011-2012 versus 86% for 2010-2011) implies that the cumulative first-year to second-year retention rate will be somewhat lower than 86%.

The trend in Calculus I GPA is slightly negative, while the Physics I GPA is essentially flat. The average composite ACT score for students in the first year of implementation is very close to the cumulative prior to implementation (27.6 prior to implementation and 27.9 the first year of implementation), but the 2011-2012 average ACT score is significantly lower (25.4), suggesting that the lower first-semester to second-semester retention is due to poorer academic aptitude/preparation, which is masking the benefits of the first-order approximation module.

Table I. Student retention and performance data before implementation (AY 2008-2009 and AY 2009-2010) and since implementation (AY 2010-2011 and AY 2011-2012).

AY	Retention	GPA		Auxiliary Data		
	1^{st}-2^{nd} Year	Calculus I	Physics I	1^{st}-2^{nd} Sem. Ret.	Average ACT	No. Students
08-09	62%	2.611	3.412	85%	27.0	26
09-10	68%	2.556	3.000	82%	28.3	22
Cumulative	65%	2.583	3.241	83%	27.6	48
10-11	86%	2.444	3.000	86%	27.9	14
11-12		2.286	3.333	74%	25.4	27
Cumulative	86%	2.444	3.083	78%	26.1	41

CONCLUSIONS

This paper has introduced a module of material on first-order approximations for freshmen engineering students and presented preliminary results on retention and success in Calculus I and Physics I. Student performance on the material itself is fair at best, with a 66% homework average. Calculus I and Physics I course grades show no improvement to date. First-to-second year retention in engineering is 86%, 21 percentage points higher after implementation than before, though the long-term improvement is likely to be lower.

To better understand the module's effectiveness, it will continue to be a part of the first-semester freshman engineering course for at least one more academic year in order to further monitor the metrics of first-to-second year retention rates and Calculus I and Physics I grades.

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APPENDIX A: STUDENT HANDOUT

Example 1: Bob, a Union Engineering student, drives from Jackson, TN, to Nashville, TN, and back, a round-trip distance of 240 miles. He travels at an average speed of 66 miles per hour. Calculate the travel time exactly and approximately.

Solution: Average speed $\bar{v}$, distance d, and time of travel t are related as:

$$\bar{v} = \frac{d}{t} \Leftrightarrow t = \frac{d}{\bar{v}}$$

Exact answer: $t = \dfrac{240\,\text{miles}}{66\,\text{miles/hr}} = 3.636363\ldots\text{hr}$

Approximate answer:

Step 0: $t = \dfrac{240\,\text{miles}}{66\,\text{miles/hr}} = \dfrac{240}{66}\text{hr}$

Step 1: $t = \dfrac{240}{60(1.1)}\text{hr}$

Step 2: $t = 4\dfrac{1}{1.1}\text{hr} = 4\dfrac{1}{1+0.1}\text{hr}$

Step 3: First-Order Approximation: $\dfrac{1}{1+0.1} \cong 1 - 0.1$

Step 4: $t \cong 4 \cdot (1-0.1)\,\text{hr} = 4 \cdot 0.9\,\text{hr}$

$$t \cong 3.6\,\text{hr}$$

Error: 1%

Table A-I. Power series expansions and first-order approximations of common functions.

Function	Power Series Expansion	First-Order Approximation
$(1+x)^m$	$1 + mx + m(m-1)\dfrac{x^2}{2} + \ldots$	$1 + mx$
$\sin x$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$	x
$\cos x$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$	1
$\tan x$	$x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \ldots$	x

e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots$	$1 + x$
$\ln(1 + x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$	x

Table A-II. Some special cases of $(1 + x)^m$ and their first-order approximations.

m	Function	First-Order Approximation
-1	$(1+x)^{-1} = \frac{1}{1+x}$	$\frac{1}{1+x} \cong 1 - x$
$\frac{1}{2}$	$(1+x)^{\frac{1}{2}} = \sqrt{1+x}$	$\sqrt{1+x} \cong 1 + \frac{x}{2}$
$-\frac{1}{2}$	$(1+x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+x}}$	$\frac{1}{\sqrt{1+x}} \cong 1 - \frac{x}{2}$
$\frac{1}{3}$	$(1+x)^{\frac{1}{3}} = \sqrt[3]{1+x}$	$\sqrt[3]{1+x} \cong 1 + \frac{x}{3}$
$-\frac{1}{3}$	$(1+x)^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{1+x}}$	$\frac{1}{\sqrt[3]{1+x}} \cong 1 - \frac{x}{3}$

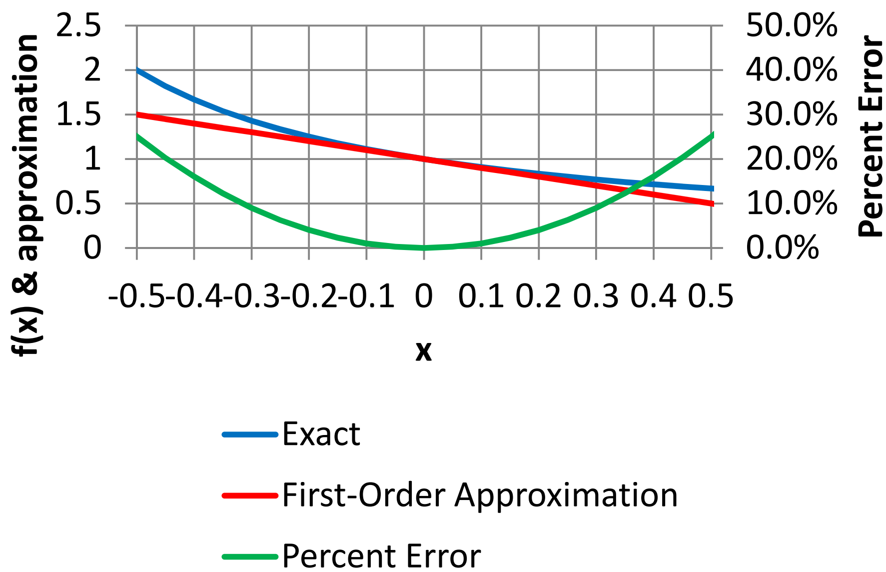

Figure 1. $\frac{1}{1+x}$, $(1 - x)$, and the percent error vs. x.

Mathematical Background

According to rules of calculus that you will learn in another term or two, a function that is differentiable around $x = 0$ can be represented by its *power series expansion*. The general form is:

$$f(x) = f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2} + ... + f^{(n)}(0) \cdot \frac{x^n}{n!} + ... \qquad \text{Eq. 1}$$

where the primed functions denote derivatives of the function $f(x)$, and $f^{(n)}(0)$ is the n^{th} derivative of f evaluated at $x = 0$. This expansion is an exact representation of the function. Approximating the function by *truncating* this series after the *first-order* term, the x^1 term, results in a *first-order* or *linear approximation* for $f(x)$.

$$f(x) \cong f(0) + f'(0) \cdot x$$

It is called "first-order," because it is a first-order polynomial, and "linear" because it is a line. This approximation is valid for "small" values of x; that is, for $|x| << 1$. When the magnitude of x is much less than 1, the higher-order terms (higher order than the "zeroth" and first-order terms) in the expansion are much, much less than 1 in magnitude and contribute very little to the total value, so the error introduced by omitting them is acceptable in some situations.

Example 2 On Bob's road trip, he consumes 8.4 gallons of fuel. What was his fuel mileage in miles per gallon (mpg)? Calculate it exactly and approximately using a first-order approximation.

Solution:

Exact: $\text{Fuel mileage} = \dfrac{240\,\text{miles}}{8.4\,\text{gal}} = 28.571\,\text{mpg}$

Approximate:

$$\text{Fuel mileage} = \frac{240}{8+0.4}$$

$$= \frac{240}{8 \cdot (1 + 0.4/8)} = \frac{240}{8} \cdot \frac{1}{1+0.05}$$

$$= 30 \cdot \frac{1}{1+0.05} \cong 30 \cdot (1 - 0.05)$$

$$\cong 30 \cdot 0.95$$

At this point, we could calculate this with our calculator or pencil and paper to be: **28.5 mpg**

Alternatively, you could observe that this is 5% less than 30, and 5% of 30 is 1.5 (half of 3 which is 10%), so the approximation is: 30-1.5 = 28.5 mpg.

Error: 0.25% -- Excellent!

Example 3 Calculate $\sqrt{10}$ exactly and approximately using a first-order approximation.

Solution:

Exact (calculator): $\sqrt{10} = 3.16227766$

Approximate: $\sqrt{10} = \sqrt{9+1} = \sqrt{9 \cdot \left(1 + \dfrac{1}{9}\right)}$

$$\sqrt{10} = 3\sqrt{1+\frac{1}{9}} = 3\cdot\left(1+\frac{1}{9}\right)^{\frac{1}{2}}$$

$$\sqrt{10} \cong 3\cdot\left(1+\frac{1}{2\cdot 9}\right) = 3\cdot 1.05555$$

$$\sqrt{10} \cong 3.16666$$

Error: 0.14% -- Excellent!

If we had further approximated $\left(1+\frac{1}{2\cdot 9}\right) \cong 1.05$, we would have obtained

$$\sqrt{10} \cong 3.15$$

with an error of 0.39%, still very good.

Example 4 Suppose there is a culture of cells that doubles in number every 10 hours. Given that there are 1000 cells initially, how many cells will there be after 2 hours? Calculate the number exactly and approximately using a first-order approximation.

Solution:

Population $P(t)$: $P(t) = 1000\cdot 2^{t/10}$

where t is in hours

Exact (calculator): $P(2) = 1000\cdot 2^{0.2} = 1149$

Approximate: $P(2) = 1000\cdot 2^{0.2} = 1000\cdot\left(e^{\ln 2^{0.2}}\right)$

$$P(2) = 1000\cdot\left(e^{0.2\cdot\ln 2}\right)$$

First-order approximation: $e^x \cong 1+x$

$$e^{0.2\cdot\ln 2} \cong 1+0.2\cdot\ln 2$$

$$P(2) \cong 1000\cdot(1+0.2\cdot\ln 2) = 1139$$

Error: 0.87% -- Excellent!

Example 5 The pH scale is used in chemistry to measure acidity or basicity. The definition of pH is:

$$\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$$

Where $[H^+]$ is the concentration of hydrogen ions in moles per liter. A certain brand of vinegar has a nominal hydrogen ion concentration of 6.3×10^{-3} mol/L.

a. Calculate the pH of the vinegar with the nominal concentration.
b. Calculate the pH if the concentration is 5% above nomimal. Calculate both exactly and via a first-order approximation.

Solution:

a. $\mathrm{pH} = -\log_{10} 6.3\times10^{-3} = 2.201$

b. Exact: $\mathrm{pH} = -\log_{10}\left(6.3\times10^{-3}\cdot 1.05\right) = 2.179$

Approximate:

$$\text{pH} = -\log_{10}\left(6.3\times10^{-3}\cdot 1.05\right) = -\log_{10} 6.3\times10^{-3} - \log_{10} 1.05$$

$$\log_{10} 1.05 = \frac{\ln 1.05}{\ln 10} = \frac{\ln\left(1+0.05\right)}{\ln 10}$$

$$\log_{10} 1.05 \cong \frac{0.05}{\ln 10} = 0.0217$$

$$\text{pH} \cong -\log_{10} 6.3\times10^{-3} - 0.0217 = 2.179$$

Error: 0% to 3 decimal places

Example 6 A team of engineering students sets out to measure the height of the goal posts on a local high school football field. They do this not by climbing the goal posts and measuring the height directly but by measuring the angle from the ground to the top of the goal posts as seen from the opposite goal line. This angle is measured to be 5.2°. What is the height? Calculate exactly and approximately using a first-order approximation. Note that the field is 300 feet long, and the end zone is 30 feet long.

Solution:

Exact: $$\tan 5.2^\circ = \frac{H}{330}$$

$$H = 330\tan 5.2^\circ \text{ ft} = 30.0 \text{ ft}$$

Approximate: $$5.2° = 5.2^\circ = 5.2^\circ \cdot \frac{\pi}{180^\circ} \text{ rad} = 0.0908 \text{ rad}$$

$$\tan x \cong x \text{ for small } x$$

$$H = 330\tan 0.0908 \text{ ft} \cong 330\cdot 0.0908 \text{ ft}$$

$$H \cong 29.96 \text{ ft} \cong 30.0 \text{ ft}$$

Error: 0% to 1 decimal place

APPENDIX B: HOMEWORK ASSIGNMENT

Use Excel to create any plots.

1. To investigate the approximation

$$\sin x \cong x \quad \text{for small x}$$

plot sin x vs. x and x vs. x for $-1 \le x \le 1$ on the same graph. Reference these to the left vertical axis. On the same graph, plot the percent error of the approximation vs. x, referenced to the right vertical axis. Recall that

$$\text{Percent error} = \left|\frac{\sin x - x}{\sin x}\right| \times 100\%$$

2. To investigate the approximation

$$\tan x \cong x \quad \text{for small } x$$

plot tan x vs. x and x vs. x for $-1 \le x \le +1$ on the same graph. Reference these to the left vertical axis. On the same graph, plot the percent error of the approximation vs. x, referenced to the right vertical axis. Recall that

$$\text{Percent error} = \left|\frac{\tan x - x}{\tan x}\right| \times 100\%$$

3. A square has an area of 90 cm^2.
   a. Calculate the length of a side exactly.
   b. Calculate an approximate length of a side using a first-order approximation.
   c. Calculate the percent error of the approximate value.
4. A cube has a volume of 1100 cm^3.
   a. Calculate the length of a side exactly.
   b. Calculate an approximate length of a side using a first-order approximation.
   c. Calculate the percent error of the approximate value.
5. In a "resistor-capacitor" ("RC") circuit, the capacitor is initially discharged with a voltage across it of zero. At time $t = 0$ s, the switch closes, and the capacitor charges up. For $R = 10^6$ Ohms and $C = 10^{-6}$ F, the voltage across the capacitor for $t \ge 0$ s is given by:

$$v(t) = 10 \cdot (1 - e^{-t}) \text{ V}$$

Find the value of the voltage at $t = 0.1$ s, 0.2 s, and 0.5 s both exactly and using a first-order approximation.

6. Use the fact that $\sin\frac{\pi}{6} = \frac{1}{2}$ along with the first-order approximation for sin x to obtain an approximate value of π.

Comment and Hint: You might observe that $\pi/6$ is not "much less than" one and therefore not expect the resulting estimate to be very good. However, due to the particular characteristics of the sine function, such as the next highest term in its expansion being 3rd order rather than 2nd order, the percent error of the estimate is only 4.5%. By the way, this value is an ancient estimate of π. The Bible alludes to it in 1 Kings 7:23.

7. A certain two-terminal electronic device has the following "I-V relationship" (that is, the current through the device is related to the voltage across it):

$$V = 0.05 \cdot \ln\left(10^9 \cdot I\right) \text{V}$$

The current I is in amperes (A), and the voltage V is in volts (V). The device is operated at first with a current of 0.001 A.
   a. What is the voltage V initially (that is, for current $I = 0.001$ A)?
   b. What is the voltage V if the current increases from 0.001 A to 0.0011 A? Calculate it exactly and by means of a first order approximation. You will need to do a little algebraic manipulation and use your knowledge of logarithms in order to apply the first-order approximation for $\ln(1 + x)$.
   c. What is the voltage if instead the current decreases from 0.001 A to 0.0009 A? Calculate it exactly and by means of a first order approximation.

Extra Credit Find an approximate value of π using the fact that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and the second- and fourth-order approximations for cos x.