Evaluation of Bessel Functions Using a Computer Program

P. S. Yeh, Ph.D. [1]

Abstract – In cylindrical coordinate, there are two types of Bessel functions. These functions are the Bessel function and the modified Bessel function. Both functions are expressed mathematically by infinite power series, and each one consists of different orders, beginning with the zero-order, and then the first order, the second order, and so on. The Bessel function is the solution of the Bessel differential equation, which is a linear, second-order ordinary differential equation. Similarly, the modified Bessel function is the solution of modified Bessel differential equation. The difference between these two differential equations is the signs of the non-differential terms. The applications of Bessel functions are in the scientific areas of elasticity, electrical field theory, aerodynamic flutter analysis, fluid flow, and heat transfer by conduction. An executable computer program has been developed in this study for the numerical computation of the Bessel function and the modified Bessel function. This program is called BESSEL.EXE. It is distributed free by contacting the author through his e-mail address.

Keyword: Bessel functions, modified Bessel functions, computer program.

INTRODUCTION

The Bessel differential equation is an ordinary linear differential equation given by the following form [Bronshtein,1], [Carslaw, 2], [Morse, 3]:

$$r^2\frac{d^2T}{dr^2}+r\frac{dT}{dr}+(r^2-m^2)T=0 \tag{1}$$

Where m is assumed to be a real and positive integer constant. T is the dependent variable such as the temperature, and r is the independent variable such as the radial coordinate in the cylindrical system. The first independent solution of Equation 1 is given by the following equation, which is known as the Bessel function of the first kind of order m [Bronshtein, 1], [Carslaw, 2], [Morse, 3], [Schneider, 4]:

$$J_m(r)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{r}{2})^{m+2n}}{n!(m+n)!} \qquad m=0,1,2,3,\cdots \tag{2}$$

The second independent solution of Equation (1), which is known as the Bessel function of the second kind of order m, is given as follows [Bronshtein, 1], [Carslaw, 2], [Morse, 3], [Schneider, 4], [Yeh, 5]:

[1] Professor Emeritus of Engineering, Jacksonville State University, 205 Macon Drive SE, Jacksonville, AL 36265. E-mail: psyeh@jsu.edu

$$Y_m(r)=\frac{1}{p}\{2J_m(r)[\ln(\frac{r}{2})+0.5772157]-\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{r}{2})^{m+2n}}{n!(m+n)!}[\sum_{p=1}^{m+n}p^{-1}+\sum_{p=1}^{n}p^{-1}]-\sum_{n=0}^{m-1}(\frac{r}{2})^{-m+2n}\cdot\frac{(m-n-1)!}{n!}\}$$

$$Where\ m=0,1,2,3,\ldots\ and\ \ for\ n=0,\ \ replace\ \ \sum_{p=1}^{m+n}p^{-1}+\sum_{p=1}^{n}p^{-1}\quad by\quad \sum_{p=1}^{m}p^{-1} \tag{3}$$

The second type of Bessel differential equation, known as the modified Bessel differential equation, is shown in the following [Bronshtein, 1], [Carslaw, 2], [Morse, 3], [Schneider, 4], [Yeh, 5], [Hilderbrand, 7]:

$$r^2\frac{d^2T}{dr^2}+r\frac{dT}{dr}-(r^2+m^2)T=0 \tag{4}$$

The two independent solutions to this differential equation are known as the modified Bessel function of the first kind of order m, and the modified Bessel function of the second kind of order m, respectively. These two solutions are given below [Bronshtein, 1], [Carslaw, 2], [Morse, 3], [Schneider, 4], [Yeh, 5], [Duffy, 7], [Hilderbrabd, 8]:

The modified Bessel function of the first kind of order m:

$$I_m(r)=\sum_{n=0}^{\infty}\frac{(\frac{r}{2})^{m+2n}}{n!(m+n)!}\quad m=0,1,2,3,\cdots \tag{5}$$

The modified Bessel function of the second kind of order m:

$$K_m(r)=(-1)^{m+1}I_m(r)[\ln(\frac{r}{2})+0.5772157]+\frac{(-1)^m}{2}\sum_{n=0}^{\infty}\frac{(\frac{r}{2})^{m+2n}}{n!(m+n)!}[\sum_{p=1}^{m+n}p^{-1}+\sum_{p=1}^{n}p^{-1}]+\frac{1}{2}\sum_{n=0}^{m-1}(-1)^n\cdot$$

$$\cdot(\frac{r}{2})^{-m+2n}\cdot\frac{(m-n+1)!}{n!}\qquad m=0,1,2,3,\ldots \tag{6}$$

$$Where\ \ for\ n=0,\ \ replace\ \ \sum_{p=1}^{m+n}p^{-1}+\sum_{p=1}^{n}p^{-1}\quad by\quad \sum_{p=1}^{m}p^{-1}$$

Notice that all solutions to the Bessel and modified Bessel differential equations, as represented by Equations (2), (3), (5) and (6), are expressed in terms of infinite power series.

On the applications of the Bessel functions and the modified Bessel functions, many technical books have presented these functions either in graphical format or in numerical tables, or in both options. However, no record on the
availability of a computer program can be found by the present author. A summary of information on the Bessel functions and the modified Bessel functions is given in Table 1.

Author or Authors	Bessel Functions				Modified Bessel Functions			
	Graphs	Range of R	Tables	Range of R	Graphs	Range of R	Tables	Range of R
Bronshtein and Somendyayev	Jo, J1, J2, J3	0 to 14	Jo, J1	0 to 10	Io, I1	0 to 3	Io, I1	0 to 10
Carslaw and Yaeger	None		None		None		None	
Morse and Feshbach	None None		Jo, J1, J2 Yo, Y1, Y2	0 to 8.0 0 to 8.0	None None		Io, I1, I2 None	0 to 8.0
Schneider	Jo, J1 Yo, Y1	0 to 12 0 to 12	Jo, J1 Yo, Y1	0 to 15 0 to 14.9	Io, I1 Ko, K1	0 to 3.0 0 to 3	Io, I1 Ko, K1	0 to 5.9 0 to 3.9
Duffy	Jo, J1, J2, J3 Yo, Y1, Y2, Y3	0 to 8.0 0 to 8.0	None None		Io, I1, I2, I3 Ko, K1, K2, K3	0 to 3.0 0 to 3.0	None None	
Hilderbrand	Jo, J1 Yo, Y1	0 to 10.0 0 to 10.0	None None		Io Ko	0 to 6.0 0 to 6.0	None None	
Yeh	Jo to J3 Yo to Y3	0 to 13.0 0 to 13.0	Jo to J3 Yo to Y3	0 to 13.0 0 to 13.0	Io to I3 Ko to K3	0 to 5.0 0 to 5.0	Io to I3 Ko to K3	0 to 13.0 0 to 13.0

Table 1. Comparison of Graphs and Tables for Bessel and Modified Bessel Functions

Computer Program

The generalized flowchart of the computer program, which was written in FORTRAN [Yeh, 5], [Yeh, 6], is shown in the following as Figure 1:

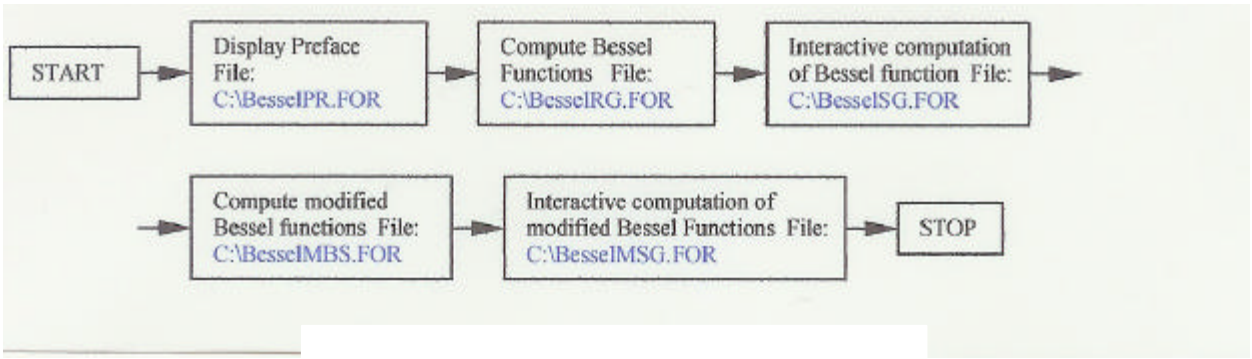

Figure 1. Flowchart for the Computer Program

Note that a total of five files are generated from the computer program. These files are stored in the C: drive, that is, the hard disk drive. Two of the five files, namely, BesselRG.FOR and BesselMBS.FOR, are in the familiar form of numerical tables. However, compared to most of the existing tables in many of the published books, the upper range of the independent variable r (or R) in the present study is higher, which is 13.10 for the Bessel functions, and 13.00 for the modified Bessel functions. Through the use of a statistical or a graphical software, such as the Microsoft Excel, these tables can be presented in graphical forms, as it will be shown in a later section.

Presentation of Results

A. The content of the computer file BesselPR.FOR is shown in Figure 2. It provides a simple description of the program, and an instruction on how to use the program interactively.

```
This program calculates Bessel Functions of the
First kind and the second kind, J and Y, and modified
Bessel functions of the first and the second kind, I and K.
Each kind contains orders from zero to three, i.e., Jo. J1,
J2, J3, Yo, Y1, Y2, Y3, Io, I1, I2, I3, and Ko, K1, K2,
K3. The results are in the form of a numerical tables,
with an increment of 0.10 in the independent variable R.
The tables are saved in C-drive, with file names as
TABLEBRG.FOR for the Bessel functions, and TABLEMBS.FOR
for the modified Bessel functions. Using Microsoft Excel,
these tables can be plotted in graphical forms.

To calculate Bessel functions and modified Bessel
functions for a given value of R, key in the value of R
and press ENTER (R is between 0.00 and 13.10 for the
Bessel functions, and between 0.00 and 5.00 for the
modified Bessel functions. To terminate the calculation,
key in -1.0 and press ENTER. The results from each of
these calculations are stored in a file in the C-drive,
namely BesselSG.FOR for the Bessel functions, and
BesselMSG.FOR for the modified Bessel functions.
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Figure 2. The Computer File BesselPF.FOR

B. For the output data file BesselRG.FOR, the numerical table consists of two pages. A portion of the first page is shown in Figure 3. The increment of the independent variable r (or R) is 0.10, and the range is from 0.00 to 13.10. The order of the Bessel function is from zero to three, which is more than many of the tables in the existing published books. Through the use of Microsoft Excel, this table can be represented in a graphical form, as shown in Figure 4 and Figure 5. The Bessel functions of the first kind possess a finite numerical value at R=0, which is either 1.00 or 0.00. While the Bessel functions of the second kind all approach negative infinite as R approaches zero. As the value of R increases, all Bessel functions display the characteristic of oscillating waves and at the same time decreasing values, that is, a damping effect.

```
THE ROOTS OF THE ZERO-ORDER BESSEL FUNCTION OF THE FIRST KIND, Jo(R):
     2.405    5.520    8.654   11.792   14.931   18.071   21.212   24.352
    27.493   30.635   33.776   36.917   40.058   43.200   46.341   49.483
    52.624   55.766   58.907   62.048

    R     J0      J1      J2      J3       Y0       Y1         Y2         Y3
 ------------------------------------------------------------------------------
   .00  1.00000  .00000  .00000  .00000 -99999.0 -99999.0   -99999.0    -99999.0
   .10   .99750  .04994  .00125  .00002 -1.53424 -6.45895 -127.64480 -5099.33200
   .20   .99002  .09950  .00498  .00017 -1.08111 -3.32382  -32.15714  -639.81900
   .30   .97763  .14832  .01117  .00056  -.80727 -2.29310  -14.48009  -190.77480
   .40   .96040  .19603  .01973  .00132  -.60602 -1.78087   -8.29833   -81.20247
   .50   .93847  .24227  .03060  .00256  -.44452 -1.47147   -5.44137   -42.05949
   .60   .91200  .28670  .04367  .00440  -.30851 -1.26039   -3.89279   -24.69157
   .70   .88120  .32900  .05879  .00693  -.19066 -1.10325   -2.96148   -15.81947
   .80   .84629  .36884  .07582  .01025  -.08680  -.97814   -2.35856   -10.81464
   .90   .80752  .40595  .09459  .01443   .00563  -.87313   -1.94591    -7.77536
  1.00   .76520  .44005  .11490  .01956   .08826  -.78121   -1.65068    -5.82152
  1.10   .71962  .47090  .13656  .02569   .16216  -.69812   -1.43147    -4.50723
  1.20   .67113  .49829  .15935  .03287   .22808  -.62114   -1.26331    -3.58990
```

```
1.30   .62009  .52202  .18303  .04114   .28654  -.54852   -1.13041    -2.92967
1.40   .56686  .54195  .20736  .05050   .33790  -.47915   -1.02239    -2.44197
1.50   .51183  .55794  .23209  .06096   .38245  -.41231    -.93219    -2.07354
1.60   .45540  .56990  .25697  .07252   .42043  -.34758    -.85490    -1.78967
1.70   .39798  .57777  .28174  .08515   .45203  -.28473    -.78700    -1.56704
1.80   .33999  .58152  .30614  .09880   .47743  -.22366    -.72595    -1.38955
1.90   .28182  .58116  .32993  .11342   .49682  -.16441    -.66988    -1.24586
2.00   .22389  .57672  .35283  .12894   .51038  -.10703    -.61741    -1.12778
2.10   .16661  .56829  .37462  .14528   .51829  -.05168    -.56751    -1.02930
2.20   .11036  .55596  .39506  .16233   .52078   .00149    -.51943     -.94591
2.30   .05554  .53987  .41391  .17998   .51808   .05228    -.47262     -.87422
2.40   .00251  .52019  .43098  .19811   .51041   .10049    -.42667     -.81161
2.50  -.04838  .49709  .44606  .21660   .49807   .14592    -.38134     -.75606
2.60  -.09680  .47082  .45897  .23529   .48133   .18836    -.33644     -.70596
2.70  -.14245  .44160  .46956  .25405   .46050   .22763    -.29189     -.66006
2.80  -.18504  .40971  .47769  .27270   .43592   .26355    -.24767     -.61736
2.90  -.22431  .37543  .48323  .29109   .40791   .29594    -.20382     -.57706
3.00  -.26005  .33906  .48609  .30906   .37685   .32467    -.16040     -.53854
3.10  -.29206  .30092  .48621  .32644   .34310   .34963    -.11754     -.50129
3.20  -.32019  .26134  .48353  .34307   .30705   .37071    -.07536     -.46491
3.30  -.34430  .22066  .47803  .35877   .26909   .38785    -.03403     -.42910
3.40  -.36430  .17923  .46972  .37339   .22962   .40102     .00628     -.39363
3.50  -.38013  .13738  .45863  .38677   .18902   .41019     .04537     -.35834
3.60  -.39177  .09547  .44481  .39876   .14771   .41539     .08306     -.32310
3.70  -.39923  .05383  .42833  .40922   .10607   .41667     .11915     -.28786
3.80  -.40256  .01282  .40930  .41803   .06450   .41411     .15345     -.25259
3.90  -.40183 -.02724  .38786  .42504   .02338   .40782     .18576     -.21729
4.00  -.39715 -.06604  .36413  .43017  -.01694   .39793     .21590     -.18202
4.10  -.38867 -.10327  .33829  .43331  -.05609   .38459     .24370     -.14684
4.20  -.37656 -.13865  .31054  .43439  -.09375   .36801     .26900     -.11183
```

Figure 3. The Output Data File BesselRG.FOR

C. The file BesselSG.FOR is shown in Figure 6. For the interactive computation, the input value for r (or R) can be in any arbitrary sequence, as long as it is within the range of 0.00 to 13.10. For each value of r, eight values of the Bessel function are evaluated.

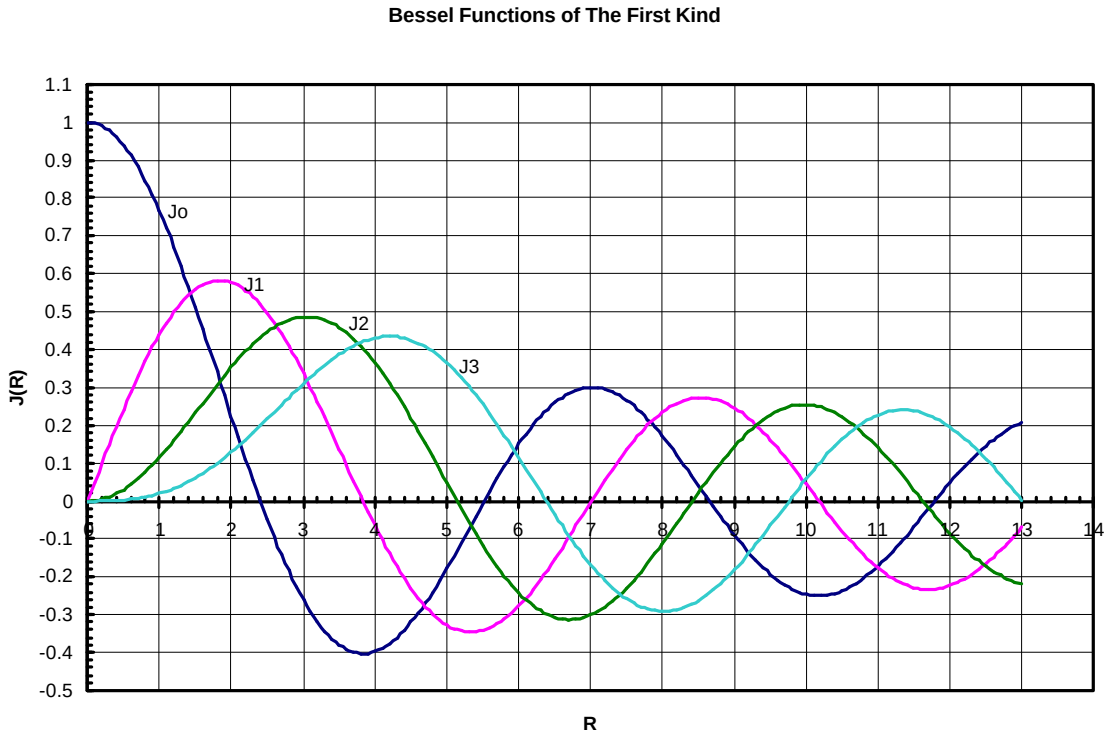

Figure 4. Graphical Presentation of the Bessel Function of the First Kind

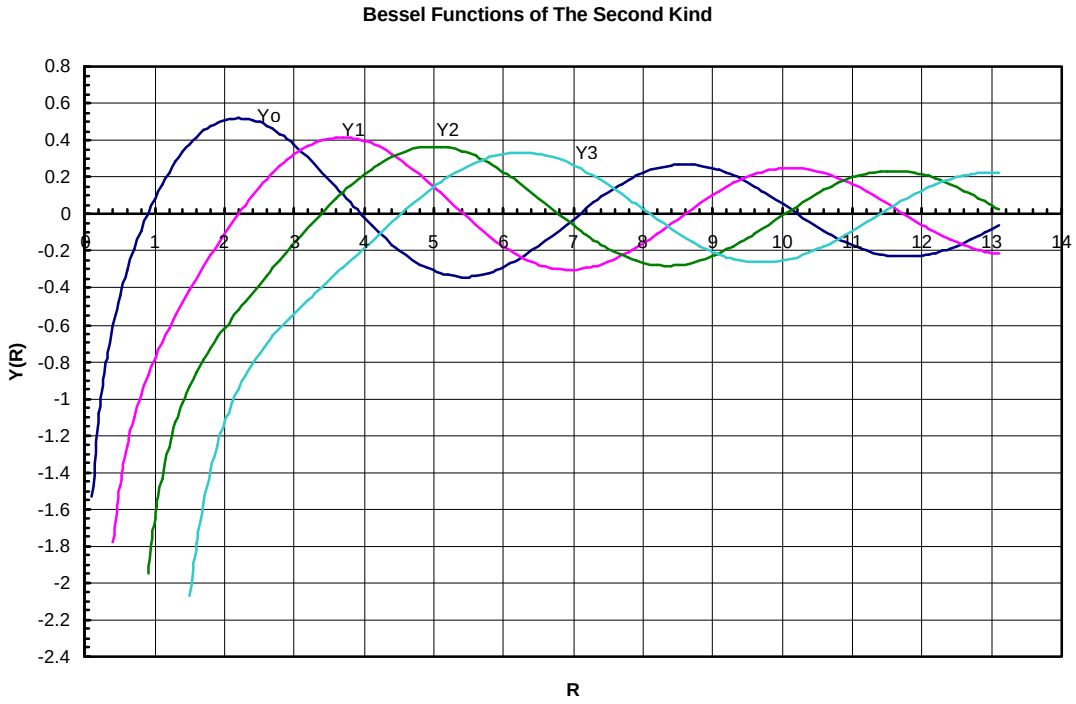

Figure 5. Graphical Presentation of the Bessel Function of the Second Kind

```
R=      .1000
 Jo=       .99750  J1=       .04994  J2=       .00125  J3=       .00002
 Yo=     -1.53424  Y1=     -6.45895  Y2=   -127.64480  Y3=  -5099.33200

 R=     1.0000
 Jo=       .76520  J1=       .44005  J2=       .11490  J3=       .01956
 Yo=       .08826  Y1=      -.78121  Y2=     -1.65068  Y3=     -5.82152

 R=     2.0000
 Jo=       .22389  J1=       .57672  J2=       .35283  J3=       .12894
 Yo=       .51038  Y1=      -.10703  Y2=      -.61741  Y3=     -1.12778

 R=     3.0000
 Jo=      -.26005  J1=       .33906  J2=       .48609  J3=       .30906
 Yo=       .37685  Y1=       .32467  Y2=      -.16040  Y3=      -.53854

 R=     4.0000
 Jo=      -.39715  J1=      -.06604  J2=       .36413  J3=       .43017
 Yo=      -.01694  Y1=       .39793  Y2=       .21590  Y3=      -.18202

 R=     5.0000
 Jo=      -.17760  J1=      -.32758  J2=       .04657  J3=       .36483
 Yo=      -.30852  Y1=       .14786  Y2=       .36766  Y3=       .14627

 R=     6.0000
 Jo=       .15065  J1=      -.27668  J2=      -.24287  J3=       .11477
 Yo=      -.28819  Y1=      -.17501  Y2=       .22986  Y3=       .32825

 R=     7.0000
 Jo=       .30008  J1=      -.00468  J2=      -.30142  J3=      -.16756
 Yo=      -.02595  Y1=      -.30267  Y2=      -.06053  Y3=       .26808
```

Figure 6. The content of the File BesselSG.FOR

D. The data file BesselMBS.FOR contains the numerical table for the modified Bessel functions, which extends over two pages. A portion of the first page is shown in Figure 7. The Microsoft Excel graphical display is shown in Figure 8. At R=0, the numerical value of the modified Bessel function of the first kind of zero order, that is, I_0 , is 1.00, while the values of all higher order functions, such as I_1 , I_2 , and I_3 , are 0.00. The values of these functions increase as the value of R increases. On the other hand, for the modified Bessel functions of the second kind, such as K_0 , K_1 , K_2 , and K_3 , the numerical values approach positive infinite as the value of R approaches zero. These functions all become very small in values as the value of R becomes large.

R	I0	I1	I2	I3	K0	K1	K2	K3
.00	1.00000	.00000	.00000	.00000	99999.0	99999.0	99999.0	99999.0
.10	1.00250	.05006	.00125	.00002	2.42707	9.85384	199.50400	7990.01200
.20	1.01003	.10050	.00502	.00017	1.75270	4.77597	49.51243	995.02450
.30	1.02263	.15169	.01133	.00057	1.37246	3.05599	21.74574	292.99910
.40	1.04040	.20403	.02027	.00135	1.11453	2.18435	12.03630	122.54740
.50	1.06348	.25789	.03191	.00265	.92442	1.65644	7.55018	62.05791
.60	1.09205	.31370	.04637	.00460	.77752	1.30283	5.12030	35.43820
.70	1.12630	.37188	.06379	.00737	.66052	1.05028	3.66133	21.97216
.80	1.16651	.43286	.08435	.01110	.56535	.86178	2.71980	14.46078
.90	1.21299	.49713	.10826	.01597	.48673	.71653	2.07903	9.95665
1.00	1.26607	.56516	.13575	.02217	.42102	.60191	1.62484	7.10126
1.10	1.32616	.63749	.16709	.02989	.36560	.50976	1.29244	5.20954
1.20	1.39373	.71468	.20260	.03936	.31851	.43459	1.04283	3.91069
1.30	1.46928	.79733	.24262	.05081	.27825	.37255	.85140	2.99223
1.40	1.55340	.88609	.28755	.06452	.24365	.32084	.70199	2.32653
1.50	1.64672	.98167	.33783	.08077	.21381	.27739	.58366	1.83380
1.60	1.74998	1.08481	.39397	.09989	.18795	.24063	.48875	1.46250
1.70	1.86397	1.19635	.45650	.12223	.16550	.20936	.41180	1.17832
1.80	1.98956	1.31717	.52604	.14819	.14593	.18262	.34885	.95784
1.90	2.12774	1.44824	.60327	.17820	.12885	.15966	.29691	.78473
2.00	2.27959	1.59064	.68895	.21274	.11389	.13987	.25376	.64739
2.10	2.44628	1.74550	.78390	.25235	.10078	.12275	.21768	.53738
2.20	2.62914	1.91409	.88906	.29763	.08927	.10790	.18736	.44855
2.30	2.82961	2.09780	1.00543	.34922	.07914	.09498	.16173	.37626
2.40	3.04926	2.29812	1.13415	.40787	.07022	.08372	.13999	.31704
2.50	3.28984	2.51672	1.27647	.47437	.06235	.07389	.12146	.26823
2.60	3.55327	2.75538	1.43374	.54963	.05540	.06528	.10562	.22777
2.70	3.84165	3.01611	1.60750	.63463	.04926	.05774	.09202	.19407
2.80	4.15730	3.30105	1.79940	.73048	.04382	.05111	.08033	.16587
2.90	4.50275	3.61261	2.01129	.83841	.03901	.04529	.07024	.14217
3.00	4.88079	3.95337	2.24521	.95975	.03474	.04016	.06151	.12217
3.10	5.29449	4.32620	2.50339	1.09602	.03096	.03563	.05394	.10524
3.20	5.74720	4.73425	2.78830	1.24888	.02759	.03164	.04737	.09086
3.30	6.24263	5.18095	3.10265	1.42016	.02461	.02812	.04165	.07860
3.40	6.78481	5.67010	3.44945	1.61191	.02196	.02500	.03666	.06813
3.50	7.37820	6.20583	3.83201	1.82639	.01960	.02224	.03231	.05916
3.60	8.02767	6.79271	4.25395	2.06610	.01750	.01980	.02850	.05146
3.70	8.73861	7.43574	4.71929	2.33380	.01563	.01763	.02516	.04483
3.80	9.51688	8.14041	5.23245	2.63257	.01396	.01571	.02223	.03911
3.90	10.36894	8.91278	5.79829	2.96581	.01248	.01400	.01966	.03416
4.00	11.30190	9.75945	6.42218	3.33727	.01116	.01248	.01740	.02989
4.10	12.32355	10.68773	7.11003	3.75111	.00998	.01114	.01541	.02617
4.20	13.44244	11.70560	7.86834	4.21194	.00892	.00994	.01366	.02295
4.30	14.66795	12.82187	8.70429	4.72486	.00799	.00887	.01211	.02014
4.40	16.01041	14.04620	9.62577	5.29549	.00715	.00792	.01075	.01770
4.50	17.48114	15.38920	10.64150	5.93008	.00640	.00708	.00954	.01556
4.60	19.09259	16.86253	11.76105	6.63553	.00573	.00632	.00849	.01370
4.70	20.85841	18.47903	12.99499	7.41946	.00514	.00565	.00754	.01207
4.80	22.79363	20.25279	14.35497	8.29032	.00459	.00505	.00670	.01064
4.90	24.91473	22.19930	15.85378	9.25743	.00412	.00453	.00596	.00939
5.00	27.23981	24.33559	17.50557	10.33112	.00369	.00404	.00531	.00829
5.10	29.78879	26.68037	19.32589	11.52280	.00331	.00362	.00473	.00733

Figure 7. The output Data File BesselMBS.FOR

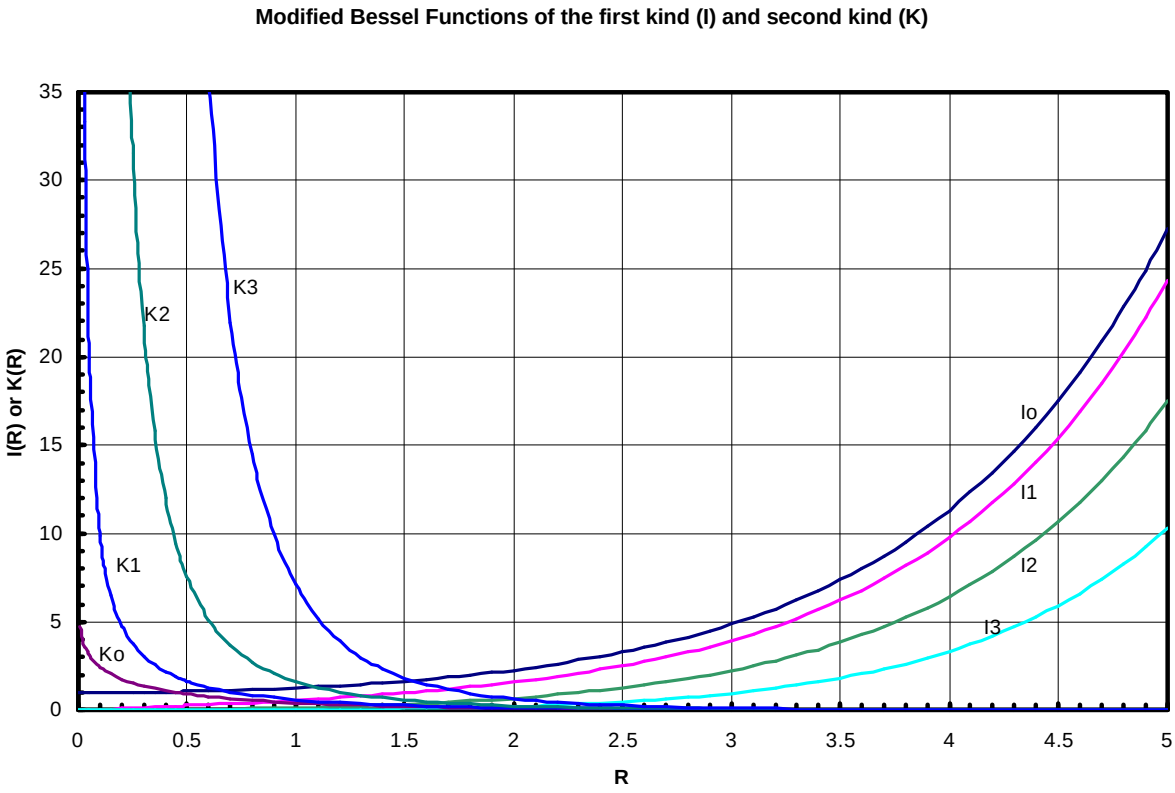

Figure 8. Graphical Presentation of the Modified Bessel Functions of the First (I) and Second (K) Kinds

E. For the interactive computation of the modified Bessel functions, the output file BesselMSG.FOR is shown in Figure 9. The file includes all input values of R and the corresponding values of the modified Bessel functions.

```
To start interactive computation of modified Bessel
 functions, key in the value of R and press ENTER. To stop
 the calculation, key in -1.0 and press ENTER. The results
 are stored in a file in the C-drive, named BesselMSG.FOR

 R=    .9000
 Io=      1.21299  I1=       .49713  I2=       .10826  I3=       .01597
 Ko=       .48673  K1=       .71653  K2=      2.07903  K3=      9.95666

 R=   6.0000
 Io=     67.23442  I1=     61.34194  I2=     46.78709  I3=     30.15054
 Ko=       .00125  K1=       .00134  K2=       .00169  K3=       .00248

 R=   3.5000
 Io=      7.37820  I1=      6.20583  I2=      3.83201  I3=      1.82639
 Ko=       .01960  K1=       .02224  K2=       .03231  K3=       .05916

 R=   7.0000
 Io=    168.59390  I1=    156.03910  I2=    124.01130  I3=     85.17548
 Ko=       .00047  K1=       .00042  K2=       .00057  K3=       .00077

 R=   1.5500
 Io=      1.69706  I1=      1.03224  I2=       .36514  I3=       .08995
 Ko=       .20042  K1=       .25826  K2=       .53366  K3=      1.63543

 R=    .0850
 Io=      1.00181  I1=       .04254  I2=       .00090  I3=       .00001
```

```
Ko=      2.58751  K1=     11.63361  K2=    276.31960  K3=  13014.91000

R=  10.0000
Io=   2815.71600  I1=   2670.98900  I2=   2281.51900  I3=   1758.38100
Ko=        .00000  K1=        .00000  K2=        .00000  K3=        .00000
```

Figure 9. The Content of the File BesselMSG.FOR

SUMMARY

For the evaluation of Bessel functions and modified Bessel functions, an executable computer program has been developed in the present study. The program is named BESSEL.EXE. A free copy of this program can be obtained from the author by contacting him through his e-mail address. No specific computer programming language compiler is required in the computer system itself, as long as the machine is IBM compatible. The program creates numerical tables for the Bessel and modified Bessel functions, respectively, for both the first kind and the second kind, for order from zero to third. The range of the independent variable can be from 0 to 13.0, and the increment is 0.10. To obtain immediate feedback from the program, it can be run interactively, by entering a numerical value of the independent variable between 0.0 and 13.00. All computed function values are stored in a file in the C: drive, and also displayed on the computer display screen. The interactive process can be repeated as many times as needed.

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P. S. Yeh

P. S. Yeh received his B.S. degree from the National Taiwan University in Taipei, Taiwan, the M.S. degree from the University of Illinois in Urbana-Champaign, Illinois, and the Ph.D. degree from Rutgers University in New Brunswick, New Jersey, all in Mechanical Engineering. He is a Professor Emeritus in Engineering at Jacksonville State University, in Jacksonville, Alabama. His major areas of teaching and research are in fluid mechanics, thermodynamics, heat transfer, computer-aided design, computer programming, acoustics, and environmental engineering.