

USFKAD: An Expert System for Partial Differential Equations

Arthur David Snider¹, Sami Kadamani²

Abstract – The execution of the solution, by the separation of variables process, of the Poisson, diffusion, and wave equations (homogeneous or nonhomogeneous) in rectangular, cylindrical, or spherical coordinate systems, with Dirichlet, Neumann, Robin, singular, periodic, or Sommerfeld boundary conditions, can be carried out in the time, Laplace, or frequency domains by a decision-tree process, using a library of eigenfunctions. We describe an expert system, USFKAD, that has been constructed for this purpose.

Keywords: Partial differential equations, symbolic computing, eigenfunctions, analytic solutions, separation of variables.

PARTIAL DIFFERENTIAL EQUATIONS IN THE ENGINEERING CURRICULUM

Every practicing engineer whose specialty involves modeling of physical phenomena, such as electromagnetic fields, temperature, sound, stress and strain, fluid flow, diffusion, etc., has to deal with the mathematical syntax of the discipline - the partial differential equation (PDE). For example, the electrical engineering undergraduate lecturer in electromagnetics, semiconductor processing, thermal issues in electronic packaging, etc. should be able to call on this mathematical concept, at least peripherally, to provide the students some familiarity with the technical issues involved in the quantitative models. However, this subject (PDEs) is vast and complicated, and compromises have to be made in incorporating it into the undergraduate's curriculum. A 2-semester course that deals honestly and rigorously with the subject is out of the question.

The compromises presently employed at undergraduate institutions are:

- (1) A short treatment of PDEs that relies completely on numerical solvers; or
- (2) A brief tutorial that covers the basics of the separation of variables technique.

Each of these is unsatisfactory. (1) is inferior to (2) because, even with the graphic capabilities of today's hardware and software, it is extremely difficult for an inexperienced undergraduate user to tell, from a vast assemblage of tabulations and graphs, how the solutions will respond to changes in the boundary conditions or the physical dimensions - issues of prime importance to engineering. For example consider the frequency of the resonant mode of a rectangular cavity with sides X, Y and Z given by $\omega = c\pi \sqrt{1/X^2 + 1/Y^2 + 1/Z^2}$. This is not a terribly complicated formula, but contemplate trying to deduce it from graphs!

The eigenfunction expansions yielded by (2) do reveal these dependencies (and are exact). The drawback of this solution procedure is the lack of time to impart expertise in its implementation except for a few elementary cases -

¹ Professor Emeritus, Dept. of Electrical Engineering, University of South Florida, Tampa FL 33620, snider@eng.usf.edu

² Professor, Department of Mathematics, Hillsborough Community College, Tampa FL 33675-5096.

rectangular geometries and ideal conductors, for instance. The electrical world of cables, motors, and antennas is replete with cylindrical and spherical devices made of lossy materials, whose analyses entail Bessel functions and transcendental eigenvalue equations. The present-day curriculum has no room for the mastery of the "special functions" that occupied the toolbox of the 1950s engineer. On the other hand, usually it is well within the capability of senior undergraduates to *observe* and *verify* most features of an eigenfunction solution expansion.

Therefore an expert system, USFKAD, for partial differential equations that can automatically cull, from a library of about 200 eigenfunctions, the assemblage constituting the solutions to explicit problems would be a powerful enabler for undergraduate engineering training:

1. It would allow engineering analysis/design to proceed efficiently without being sidetracked by concerns of mathematical solvability;
2. It would cut across many engineering disciplines;
3. It could be used to give a perspective on the separation-of-variables technique itself, by enabling "reverse-engineering" of the explicit solution formulas.

In fact it would also be a research tool that the engineer could continue to use in his professional career. Eigenfunction expansions are integral to the mode-matching procedure that is used in contemporary computational electromagnetism. And indeed, virtually every technical paper describing a new numerical solver compares its results with eigenfunction expansions, as testament to its accuracy.

PDE SOLUTION STRUCTURES

USFKAD is based on the theme that the mathematical structure, afforded by superposition, of the eigenfunction method for solving the separable PDEs of engineering can be expressed by a compact, universal, inviolate, and reasonably lucid algorithm [1]; its formidability lies only in the details of its implementation - that is, in the enormous variety of eigenfunctions that must be employed for the curvilinear geometries. Thus it is feasible to contemplate a smart computer program that exploits this structure judiciously to select, from a library of eigenfunctions, the assemblage constituting the solutions to problems with explicit initial/boundary conditions.

In the following, a list of examples of PDE solutions will be presented that progressively demonstrate the decision-tree nature of the general separation of variables procedure. (In fact, this will exemplify our thesis that by reverse-engineering explicit solution formulas one can experience a tutorial intercourse with the procedure itself.)

Example 1 Steady state heat flow in a rectangle with edge and interior heat sources (nonhomogeneous Laplace/Poisson equation in two dimensions, rectangular coordinates, Dirichlet conditions on two sides, Neumann conditions on two sides):

$$\begin{aligned}\nabla^2\Psi &= f_{\text{interior}}(x, y) \\ \Psi(0, y) &= 0, \quad \Psi(X, y) = f_{x=X}(y) \\ \frac{\partial\Psi}{\partial y}(x, 0) &= f_{y=0}(x), \quad \frac{\partial\Psi}{\partial y}(x, X) = 0\end{aligned}$$

The USFKAD solution is as follows:

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3$$

$$\Psi_1 = \sum_{\kappa_x} \cos\kappa_x x \eta_y(y; \kappa_x) A(\kappa_x), \text{ with}$$

$$\kappa_x = 0, \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \dots$$

$$\eta_y(y; \kappa_x) = \begin{cases} Y-y & \text{if } \kappa_x=0; \\ \sinh \kappa_x (Y-y) & \text{otherwise.} \end{cases}$$

$$A(\kappa_x) = \int_0^X dx \cos \kappa_x x N_{\kappa_x} M_{\kappa_x} f_{y=0}(x)$$

$$N_{\kappa_x} = \begin{cases} \frac{1}{X} & \text{if } \kappa_x=0; \\ \frac{2}{X} & \text{otherwise} \end{cases}$$

$$M_{\kappa_x} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_x=0; \\ \frac{1}{\sinh \kappa_x Y} & \text{otherwise.} \end{cases}$$

$$\Psi_2 = \sum_{\kappa_y} \sin \kappa_y y \cosh \kappa_y x A(\kappa_y) \quad , \text{ with}$$

$$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \dots$$

$$A(\kappa_y) = \int_0^Y dy \sin \kappa_y y \frac{2}{Y} M_{\kappa_y} f_{x=X}(y)$$

$$M_{\kappa_y} = \begin{cases} 0 & \text{if } \kappa_y=0; \\ \frac{1}{\kappa_y \sinh \kappa_y X} & \text{otherwise.} \end{cases}$$

$$\Psi_3 = \sum_{\kappa_x} \sum_{\kappa_y} \cos \kappa_x x \sin \kappa_y y A(\kappa_x, \kappa_y) \quad , \text{ with}$$

$$\kappa_x = 0, \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \dots$$

$$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \dots$$

$$A(\kappa_x, \kappa_y) = \int_0^X dx \int_0^Y dy \cos \kappa_x x N_{\kappa_x} \sin \kappa_y y \frac{2}{Y} \frac{f_{\text{interior}}(x, y)}{\kappa_x^2 + \kappa_y^2}$$

$$N_{\kappa_x} = \begin{cases} \frac{1}{X} & \text{if } \kappa_x=0; \\ \frac{2}{X} & \text{otherwise.} \end{cases}$$

Here we see some of the features of separation of variables:

1. The basic decomposition of the problem into three subproblems, each of which contains only one nonhomogeneous equation;
2. Each subsolution expressed as a sum of terms containing an eigenfunction factor, satisfying homogeneous boundary conditions at each end, and a “non-eigenfunction” factor, satisfying a

homogeneous boundary condition at one end only (note the exceptional form of the latter factor in Ψ_2 , and of the normalization constants for the cosine's "DC" term when $\kappa_x = 0$);

3. Expansion coefficients computed by orthogonality for boundary nonhomogeneities;
4. The construction of the Green's function out of the same eigenfunctions.

Example 2 Steady state heat flow in a cube with facial heat sources and imperfect facial insulation (homogeneous Laplace equation in three dimensions, rectangular coordinates, homogeneous Dirichlet conditions on four sides, nonhomogeneous Dirichlet condition on one side, homogeneous Robin condition on one side):

$$\nabla^2 \Psi = 0$$

$$\Psi(0, y, z) = 0, \Psi(X, y, z) = 0$$

$$\frac{\partial \Psi}{\partial z}(x, y, 0) = 0, \frac{\partial \Psi}{\partial z}(x, y, Z) = f_{z=Z}(x, y)$$

$$\Psi(x, 0, z) = 0, \frac{\partial \Psi}{\partial y}(x, Y, z) = 0$$

The USFKAD solution:

$$\Psi = \sum_{\kappa_x} \sum_{\kappa_y} \sin \kappa_x x \eta_y(y; \kappa_y) \cosh \sqrt{\kappa_x^2 + \kappa_y^2} z A(\kappa_x, \kappa_y), \text{ with}$$

$$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \dots$$

$\eta_y(y; \kappa_y) = \sin \kappa_y y$ where $\{\kappa_y\}$ are the nonzero solutions (possibly imaginary) to

$$\alpha_y \sin \kappa_y Y + \cos \kappa_y Y = 0$$

also if $\alpha_y Y + 1 = 0$ include $\kappa_y = 0$, $\eta_y(y; 0) = y$.

$$A(\kappa_x, \kappa_y) = \int_0^X dx \int_0^Y dy \sin \kappa_x x \frac{2}{x} \eta_y(y; \kappa_y) N_{\kappa_y} M_{\sqrt{\kappa_x^2 + \kappa_y^2}} f_{z=Z}(x, y)$$

$$N_{\kappa_y} = \begin{cases} \frac{3}{Y^3} & \text{if } \kappa_y = 0; \\ \frac{2}{Y + (\alpha_y + 1)/(\alpha_y^2 + \kappa_y^2)} & \text{otherwise.} \end{cases}$$

$$M_{\sqrt{\kappa_x^2 + \kappa_y^2}} = \begin{cases} 0 & \text{if } \sqrt{\kappa_x^2 + \kappa_y^2} = 0; \\ \frac{1}{\sqrt{\kappa_x^2 + \kappa_y^2} \sinh \sqrt{\kappa_x^2 + \kappa_y^2} Z} & \text{otherwise.} \end{cases}$$

This example illustrates the straightforward extension of the procedure to three dimensions and the transcendental equation that the Robin boundary condition invokes for the eigenvalues.

Example 3 Steady state heat flow in a cylindrical sector with facial heat sources (homogenous Laplace equation in the three dimensions inside a partial cylinder, nonhomogenous Dirichlet condition on the top and one flat side, homogenous Dirichlet conditions on the bottom and the curved wall, and a homogenous Neumann condition on the other flat side):

$$\begin{aligned}\nabla^2 \Psi &= 0 \\ \Psi(\rho, \theta, 0) &= \Psi(b, \theta, z) = 0, \\ \frac{\partial \Psi}{\partial \theta}(\rho, 0, z) &= 0, \quad \Psi(\rho, \Theta, z) = f_{\theta=\Theta}(z, \rho), \\ \Psi(\rho, \theta, Z) &= f_{z=Z}(\theta, \rho)\end{aligned}$$

The USFKAD solution:

$$\begin{aligned}\Psi &= \Psi_1 + \Psi_2 \\ \Psi_1 &= \sum_{\kappa_z} \int_0^\infty d\kappa_{\rho,z} \sin \kappa_z z [K_{i\kappa_{\rho,z}}(\kappa_z b) I_{i\kappa_{\rho,z}}(\kappa_z \rho) - I_{i\kappa_{\rho,z}}(\kappa_z b) K_{i\kappa_{\rho,z}}(\kappa_z \rho)] \times \cosh K_{\rho,z} \theta A(\kappa_z, \kappa_{\rho,z})\end{aligned}$$

with

$$\begin{aligned}A(\kappa_z, \kappa_{\rho,z}) &= \int_0^z dz \int_0^b d\rho \sin \kappa_z z \frac{2}{Z} [K_{i\kappa_{\rho,z}}(\kappa_z b) I_{i\kappa_{\rho,z}}(\kappa_z \rho) - I_{i\kappa_{\rho,z}}(\kappa_z b) K_{i\kappa_{\rho,z}}(\kappa_z \rho)] \frac{1}{\rho} \left[\frac{2\kappa_{\rho,z} \sinh \kappa_{\rho,z} \pi}{\pi^2 [I_{i\kappa_{\rho,z}}(\kappa_z b)]^2} \frac{1}{\cosh \kappa_{\rho,z} \Theta} f_{\theta=\Theta}(z, \rho) \right] \\ \Psi_2 &= \sum_{\kappa_\theta} \sum_{\kappa_{\rho,\theta}} \cos \kappa_\theta \theta J_{\kappa_\theta}(\kappa_{\rho,\theta} \rho) \eta_z(z; \kappa_{\rho,\theta}) A(\kappa_\theta, \kappa_{\rho,\theta})\end{aligned}$$

$$\text{with } \kappa_\theta = \frac{\pi}{2\theta}, \frac{3\pi}{2\theta}, \frac{5\pi}{2\theta}, \dots$$

For each value of $\kappa_\theta, \kappa_{\rho,\theta} = \frac{j_{\kappa_\theta, \kappa_{\rho,\theta}}}{b}$ where $\{j_{\kappa_\theta, \kappa_{\rho,\theta}}\}$ are the positive roots of $J_{\kappa_\theta}(j_{\kappa_\theta, \kappa_{\rho,\theta}}) = 0$

$$\eta_z(z; \kappa_{\rho,\theta}) = \begin{cases} z & \text{if } \kappa_{\rho,\theta} = 0; \\ \sin \kappa_{\rho,\theta} z & \text{otherwise.} \end{cases}$$

$$A(\kappa_\theta, \kappa_{\rho,\theta}) = \int_0^\Theta d\theta \int_a^b d\rho \cos \kappa_\theta \theta \frac{2}{\Theta} J_{\kappa_\theta}(\kappa_{\rho,\theta} \rho) \rho \frac{2}{b^2 J_{\kappa_\theta+1}^2(j_{\kappa_\theta, \kappa_{\rho,\theta}})} M_{\kappa_{\rho,\theta}} f_{z=Z}(\theta, \rho)$$

$$M_{\kappa_{\rho,\theta}} = \begin{cases} \frac{1}{z} & \text{if } \kappa_{\rho,\theta} = 0, \\ \frac{1}{\sin \kappa_{\rho,\theta} Z \theta} & \text{otherwise.} \end{cases}$$

This example illustrates the singular boundary condition for $\rho = 0$ and the Lebedev eigenfunctions in ρ induced by the nonhomogenous condition on the flat face for $\theta = \Theta$. The singular boundary condition includes a continuous, rather than discrete, spectrum. (Other than [2], the Lebedev expansions do not appear in any English language mathematics textbook except [3], where their correctness is betrayed by a persistent systematic error in the tabulations. Their omission is probably due to their intimidating nomenclature {note the analytic continuation of the subscript into the complex plane}. Ignoring them is, however, criminal, because {as we see} they occur in realistic problems; indeed, in the analysis of edge diffraction [2] they are crucial.)

Example 4 Transient heat flow in a rectangle with transient interior and edge heat sources (nonhomogenous diffusion equation, two dimensions, rectangular coordinates, homogenous Dirichlet conditions on two sides, homogenous Neumann condition on one side, time-dependent nonhomogenous Neumann conditions on one side):

$$\begin{aligned}\frac{\partial \Psi}{\partial t} &= \nabla^2 \Psi + f_{\text{interior}}(x, y, t) \\ \Psi(0, y) &= 0, \Psi(X, y) = 0 \\ \frac{\partial \Psi}{\partial y}(x, 0) &= 0, \frac{\partial \Psi}{\partial y}(x, Y) = f_{y=Y}(x, t)\end{aligned}$$

With $F_{\text{interior}}(x, y, s)$ and $F_{y=Y}(x, s)$ denoting the Laplace transforms of $f_{\text{interior}}(x, y, t)$ and $f_{y=Y}(x, t)$, respectively, USFKAD expresses the Laplace-transformed solution as

$$\Psi = \Psi_1 + \Psi_2$$

$$\Psi_1 = \sum_{\kappa_x} \sin \kappa_x x \cosh \sqrt{\kappa_x^2 + s} y A(s; \kappa_x)$$

$$\text{with } \kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \dots$$

$$A(s; \kappa_x) = \int_0^X dx \sin \kappa_x x \frac{2}{X} M_{\sqrt{\kappa_x^2 + s}} F_{y=Y}(x; s)$$

$$M_{\sqrt{\kappa_x^2 + s}} = \begin{cases} 0 & \text{if } \sqrt{\kappa_x^2 + s} = 0; \\ \frac{1}{\sqrt{\kappa_x^2 + s} \sinh \sqrt{\kappa_x^2 + s} Y} & \text{otherwise.} \end{cases}$$

$$\Psi_2 = \sum_{\kappa_x} \sum_{\kappa_y} \sin \kappa_x x \cos \kappa_y y A(\kappa_x, \kappa_y; s)$$

with

$$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \dots$$

$$\kappa_y = 0, \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \dots$$

$$A(\kappa_x, \kappa_y; s) = \int_0^X dx \int_0^Y dy \sin \kappa_x x \frac{2}{X} \cos \kappa_y y N_{\kappa_y} \left(\frac{F_{\text{interior}}(x, y; s) + \Psi(x, y, t=0)}{\kappa_x^2 + \kappa_y^2 + s} \right)$$

$$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise.} \end{cases}$$

This example demonstrates how the logic that solved Example 1 can be retooled to solve Laplace domain problems; the transformed PDE is equivalent to a nonhomogenous Laplace (Poisson) equation with the eigenvalues shifted and the initial condition wedded with the nonhomogeneity.

Example 5 Sound wave inside a sphere (homogenous wave equation inside a sphere, time-independent nonhomogenous Dirichlet condition on the surface):

$$\frac{\partial^2 \Psi}{\partial t^2} = \nabla^2 \Psi$$

$$\Psi(r = b, \theta, \phi, t) = f_{r=b}(\theta, \phi)$$

The USFKAD solution:

$$\Psi = \Psi_{\text{steadystate}} + \Psi_{\text{transient}}$$

$$\Psi_{\text{transient}} = \Psi_{\text{transient}\#1} + \Psi_{\text{transient}\#2}$$

$$\Psi_{\text{steadystate}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\phi, \theta) r^{\ell} A_{\ell m}$$

with

$$A_{\ell m} = \int_0^{\pi} \sin \phi d\phi \int_0^{2\pi} d\theta Y_{\ell m}^*(\phi, \theta) M_r f_{r=b}(\theta, \phi)$$

$$M_r = \frac{1}{b^{\ell}}$$

$$\Psi_{\text{transient}\#1} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p=1}^{\infty} Y_{\ell m}(\phi, \theta) j_{\ell}(\kappa_r r) \cos \kappa_r t A_{\ell m}(\kappa_r)$$

with

$$\kappa_r = s_{\ell,p} / b \text{ where } s_{\ell,p} \text{ is the } p\text{th positive zero of } j_{\ell}.$$

$$A_{\ell m}(\kappa_r) = \int_0^{\pi} \sin \phi d\phi \int_0^{2\pi} d\theta \int_0^b r^2 dr Y_{\ell m}^*(\phi, \theta) j_{\ell}(\kappa_r r) \frac{2}{b^3 j_{\ell+1}^2(s_{\ell,p})} [\Psi(\theta, \phi, r; 0) - \Psi_{\text{steadystate}}]$$

$$\Psi_{\text{transient}\#2} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p=1}^{\infty} Y_{\ell m}(\phi, \theta) j_{\ell}(\kappa_r r) \frac{\sin \kappa_r t}{\kappa_r} A_{\ell m}(\kappa_r)$$

with $\kappa_r = s_{\ell,p} / b$ where $s_{\ell,p}$ is the p th positive zero of j_{ℓ} .

$$A_{\ell m}(\kappa_r) = \int_0^{\pi} \sin \phi d\phi \int_0^{2\pi} d\theta \int_0^b r^2 dr Y_{\ell m}^*(\phi, \theta) j_{\ell}(\kappa_r r) \frac{2}{b^3 j_{\ell+1}^2(s_{\ell,p})} \frac{\partial \Psi(\theta, \phi, r; 0)}{\partial t}$$

This example demonstrates the decomposition of the solution into a steady-state component and a transient component. The time factors can be treated just like the non-eigenfunctions in the previous solutions, except that they satisfy initial conditions instead of boundary conditions.

THE STATE OF THE ART

Software for obtaining (analytic) solutions to ordinary differential equations exists in several forms, including Mathematica® and MAPLE®. It has not received universal adoption because extensive training in ordinary differential equations is already part of the required curriculum for all SMET (Science, Mathematics, Engineering, and Technology) students. For partial differential equations, MAPLE's *pdsolve* is a step in the right direction, but its arcane solution format provides little assistance for a non-expert in fitting the initial and boundary conditions that determine such dependencies. An example of its output, the electrostatic potential inside a sphere with charges distributed on the surface is displayed below. It is expressed (correctly) in terms of hyper-geometric and complex signum functions. But comparing this with the more recognizable solution display using USFKAD, one can clearly see the obvious simplification and straightforwardness of the latter:

pdsolve Solution

$$\begin{aligned}
 F(r, t, p) = & _C1 r^{(1/2\sqrt{1+4_c1})} _C3 (-2 \sin(t)^2)^{(1/2\sqrt{-c2})} \text{hypergeom}\left(\right. \\
 & \left. \left[\frac{1}{4} + \frac{1}{2}\sqrt{-c2} + \frac{1}{4}\sqrt{1+4_c1}, -\frac{1}{4}\sqrt{1+4_c1} + \frac{1}{4} + \frac{1}{2}\sqrt{-c2} \right], \left[\frac{1}{2} \right], \frac{1}{2} \cos(2t) + \frac{1}{2} \right) _C5 \\
 & \sin(\sqrt{-c2} p) / \sqrt{r} + _C1 r^{(1/2\sqrt{1+4_c1})} _C3 (-2 \sin(t)^2)^{(1/2\sqrt{-c2})} \text{hypergeom}\left(\right. \\
 & \left. \left[\frac{1}{4} + \frac{1}{2}\sqrt{-c2} + \frac{1}{4}\sqrt{1+4_c1}, -\frac{1}{4}\sqrt{1+4_c1} + \frac{1}{4} + \frac{1}{2}\sqrt{-c2} \right], \left[\frac{1}{2} \right], \frac{1}{2} \cos(2t) + \frac{1}{2} \right) _C6 \\
 & \cos(\sqrt{-c2} p) / \sqrt{r} + 2 _C1 r^{(1/2\sqrt{1+4_c1})} _C4 (-2 \sin(t)^2)^{(1/2\sqrt{-c2})} \text{csgn}(\cos(t)) \\
 & \cos(t) \text{hypergeom}\left(\left[\frac{3}{4} + \frac{1}{2}\sqrt{-c2} + \frac{1}{4}\sqrt{1+4_c1}, -\frac{1}{4}\sqrt{1+4_c1} + \frac{3}{4} + \frac{1}{2}\sqrt{-c2} \right], \left[\frac{3}{2} \right], \right. \\
 & \left. \frac{1}{2} \cos(2t) + \frac{1}{2} \right) _C5 \sin(\sqrt{-c2} p) / \sqrt{r} + 2 _C1 r^{(1/2\sqrt{1+4_c1})} _C4 \\
 & (-2 \sin(t)^2)^{(1/2\sqrt{-c2})} \text{csgn}(\cos(t)) \cos(t) \text{hypergeom}\left(\right. \\
 & \left. \left[\frac{3}{4} + \frac{1}{2}\sqrt{-c2} + \frac{1}{4}\sqrt{1+4_c1}, -\frac{1}{4}\sqrt{1+4_c1} + \frac{3}{4} + \frac{1}{2}\sqrt{-c2} \right], \left[\frac{3}{2} \right], \frac{1}{2} \cos(2t) + \frac{1}{2} \right) _C6 \\
 & \cos(\sqrt{-c2} p) / \sqrt{r} + _C2 r^{(-1/2\sqrt{1+4_c1})} _C3 (-2 \sin(t)^2)^{(1/2\sqrt{-c2})} \text{hypergeom}\left(\right. \\
 & \left. \left[\frac{1}{4} + \frac{1}{2}\sqrt{-c2} + \frac{1}{4}\sqrt{1+4_c1}, -\frac{1}{4}\sqrt{1+4_c1} + \frac{1}{4} + \frac{1}{2}\sqrt{-c2} \right], \left[\frac{1}{2} \right], \frac{1}{2} \cos(2t) + \frac{1}{2} \right) _C5 \\
 & \sin(\sqrt{-c2} p) / \sqrt{r} + _C2 r^{(-1/2\sqrt{1+4_c1})} _C3 (-2 \sin(t)^2)^{(1/2\sqrt{-c2})} \text{hypergeom}\left(\right. \\
 & \left. \left[\frac{1}{4} + \frac{1}{2}\sqrt{-c2} + \frac{1}{4}\sqrt{1+4_c1}, -\frac{1}{4}\sqrt{1+4_c1} + \frac{1}{4} + \frac{1}{2}\sqrt{-c2} \right], \left[\frac{1}{2} \right], \frac{1}{2} \cos(2t) + \frac{1}{2} \right) _C6 \\
 & \cos(\sqrt{-c2} p) / \sqrt{r} + 2 _C2 r^{(-1/2\sqrt{1+4_c1})} _C4 (-2 \sin(t)^2)^{(1/2\sqrt{-c2})} \text{csgn}(\cos(t))
 \end{aligned}$$

$$\cos(t) \operatorname{hypergeom}\left(\left[\frac{3}{4} + \frac{1}{2}\sqrt{-c_2} + \frac{1}{4}\sqrt{1+4-c_1}, -\frac{1}{4}\sqrt{1+4-c_1} + \frac{3}{4} + \frac{1}{2}\sqrt{-c_2}\right], \left[\frac{3}{2}\right], \frac{1}{2}\cos(2t) + \frac{1}{2}\right) \frac{\cos(\sqrt{-c_2} p)}{\sqrt{r}} \frac{C5}{C4} r^{(-1/2)\sqrt{1+4-c_1}} \frac{C4}{C6}$$

$$(-2\sin(t)^2)^{(1/2)\sqrt{-c_2}} \operatorname{csgn}(\cos(t)) \cos(t) \operatorname{hypergeom}\left(\left[\frac{3}{4} + \frac{1}{2}\sqrt{-c_2} + \frac{1}{4}\sqrt{1+4-c_1}, -\frac{1}{4}\sqrt{1+4-c_1} + \frac{3}{4} + \frac{1}{2}\sqrt{-c_2}\right], \left[\frac{3}{2}\right], \frac{1}{2}\cos(2t) + \frac{1}{2}\right) \frac{\cos(\sqrt{-c_2} p)}{\sqrt{r}} C6$$

USFKAD solution

$$\Psi_{steady\ state} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\phi, \theta) r^{\ell} A_{\ell m}, \text{ with } A_{\ell m} = \int_0^{\pi} \sin \phi d\phi \int_0^{2\pi} d\theta Y_{\ell m}^*(\phi, \theta) M_r f_{r=b}(\phi, \theta), \quad M_r = \frac{1}{b^{\ell}}$$

USFKAD's output is expressed in terms of familiar functions, and is completely rendered; all coefficients are specified in terms of the boundary data. However, because *pdesolve* was designed to solve a much more general class of PDEs [4], it employs very generic terminology; nonspecialist users may be hard pressed to recognize the simple factor r^{ℓ} , which is expressed (quite correctly) as a hypergeometric function, or the old-fashioned ("Associated Legendre function") expressions for the spherical harmonics $Y_{\ell, m}$. Furthermore there are "arbitrary constants" in the *pdesolve* display which the user has to fit to the boundary conditions.

THE USER INTERFACE

USFKAD is written in C++. Flow charts for its logic are given in [5]. The executable file outputs a LaTeX file containing the solution; the output, which can subsequently be font-customized by the user, must be processed by LaTeX to generate a postscript file. USFKAD prints the following queries on screen:

Select the Partial Differential Equation:

- 0 - Laplace or Poisson: Laplacian Psi + f(interior) = 0 ;
- 1 - Diffusion, Time Domain: dPsi/dt = Laplacian Psi + f(interior);
- 2 - Diffusion, s-plane: s Psi - Psi(t=0) = Laplacian Psi + F(interior);
- 3 - Wave, Time Domain: d2Psi/dt2 = Laplacian Psi + f(interior);
- 4 - Wave, s-plane: s2 Psi - sPsi(t=0) - Psi'(t=0) = Laplacian Psi + F(interior);
- 5 - Wave, Frequency Domain: - omega2 Psi = Laplacian Psi + F(interior) .

Is the PDE homogeneous (enter 0) or nonhomogeneous (enter 1)?

Enter 1, 2, or 3 for 1, 2, or 3 dimensions.

Select the Coordinate System: 0 - Rectangular, 1 - Cylindrical or Polar, 2 - Spherical.

Select the boundary condition at the lower (upper) end for the coordinate x (y, z, r, rho, theta):

- Enter 1 for Dirichlet, Homogeneous; Enter 2 for Dirichlet, Nonhomogeneous;
- Enter 3 for Neuman, Homogeneous; Enter 4 for Neuman, Nonhomogeneous;
- Enter 5 for Robin, Homogeneous; Enter 6 for Robin, Nonhomogeneous;
- Enter 7 for Periodic Boundry Conditions; Enter 8 for Singular Boundary Condition;
- Enter 9,10 for Sommerfeld Outgoing, Incoming Wave Condition.

CONCLUSIONS

USFKAD was constructed from flow charts depicting the algorithmic solution of partial differential equations; the charts are given in [5]. The formulas are mathematically rigorous, supported by [1]. The executable code has a size of about 1.4 MB, and it contains about 200 eigenfunction subroutines.

USFKAD's "coziness," the ease of moving between the time, Laplace, and Fourier domains, and the systematic incorporation of nonhomogeneities could make it the engineer's tool of choice in many situations. It is available for downloading at the first author's home page: <http://ee.eng.usf.edu/people/snider2.html>. Further details on the lexicon of the software in expressing the physical dimensions and boundary conditions appears in the accompanying files.

REFERENCES

- [1] Snider, A. D., *Partial Differential Equations: Sources and Solutions*, Prentice-Hall, Upper Saddle River NJ, 1999.
- [2] Felsen, L. B. and N. Marcuvitz, *Radiation and Scattering of Waves*, IEEE Press, Piscataway NJ, 1994.
- [3] Polianin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, CRC Press, 2002.
- [4] Cheb-Terrab, E.S. and K. von Bulow, "A Computational Approach for the Analytical Solving of Partial Differential Equations," *Computer Physics Communications*, v. 90, 1995, 102-116.
- [5] Kadamani, S. M., "USFKAD: An Expert System for Partial Differential Equations," PhD Dissertation, University of South Florida Department of Electrical Engineering, Tampa FL 33620, 2005.

Arthur David Snider

Arthur David Snider (Professor Emeritus) has served on the faculty at USF and at the MIT Draper Lab. He has authored several textbooks in applied mathematics and about 90 journal articles in mathematical modeling of electromagnetics, semiconductor devices, heat transfer; signal processing; optimization; differential equations; numerical analysis; and classical mechanics.

Sami Kadamani

Sami Kadamani has been on the faculty of Hillsborough Community College since 2003. He has recently earned his PhD in Engineering Science from the University of South Florida.