Effect of Boundary Conditions on Two-Dimensional Temperature Distribution in a Transformer

P. S. Yeh, Ph.D. [1]

Abstract - The core-and-coil assembly is the major element in a residential distribution transformer. It is the part where heat is generated, and in which the temperature needs to be analyzed. In a previous study conducted by the author [Yeh, 1], it was treated as a cylindrical shell or tube, with a metallic (i.e., iron) core placed at the axial center. In the present study, the core-and-coil assembly is assumed to be a solid cylinder, extending from the axis to the outer surface. The mathematical solution for the cylindrical tube model contains the summation of modified Bessel functions of the first kind of zero order and first order, modified Bessel functions of the second kind of zero order and first order, and the sine function. For the solid cylinder model, the solution contains the summation of modified Bessel function of the first kind of zero order and first order, and the sine function. For both models, the numerical results show that the temperature is the highest at the geometrical center of the assembly. It decreases both in the radial and axial directions. For engineering students, the numerical analysis of a partial differential equation can be performed with a packaged computer program using the finite element analysis. However, in order to see the beauty and the power of science and mathematics, it is probably best to express the solutions in a mathematical form, and then to extract the necessary numerical values from these final equations.

Keyword: Transformer, temperature, residential, boundary effect.

INTRODUCTION

For the above-ground distribution of electrical energy, the high-voltage input line is located on the top of the pole, the input or primary line for the transformer is connected directly to this top wire, as shown in Figure 1. The output or secondary lines are situated at the upper part and side of the tank. These lines provide the low voltage of 120/240 V directly to the consumers. Figure 2 depicts the idealized geometrical representation of the core-and-coil assembly, as well as the flow pattern of the cooling fluid. Note that the core-and-coil is simply modeled as a solid cylinder. No interior detail is described and this is perhaps not necessary. When electrical current flows through the coil, a small percentage of the electrical energy is lost in the form of heat. These losses consist of resistance loss in the windings, hysteresis loss in the core, and eddy currents in the core. The total loss is usually less than 10% of the total energy supplied. Due to these energy losses, the temperature in the assembly increases in comparison with its ambient, and if sufficient provision is provided for the continuous dissipation of heat, a steady state temperature distribution will eventually be reached.

TWO DIMENSIONAL ANALYSIS

During the steady state operation of transformer, heat is generated within the coil assembly. Since the coil carries electrical current, and the transmission of an electrical current is an irreversible process, a small fraction of the

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Figure 1. A 25-kVA Distribution Transformer

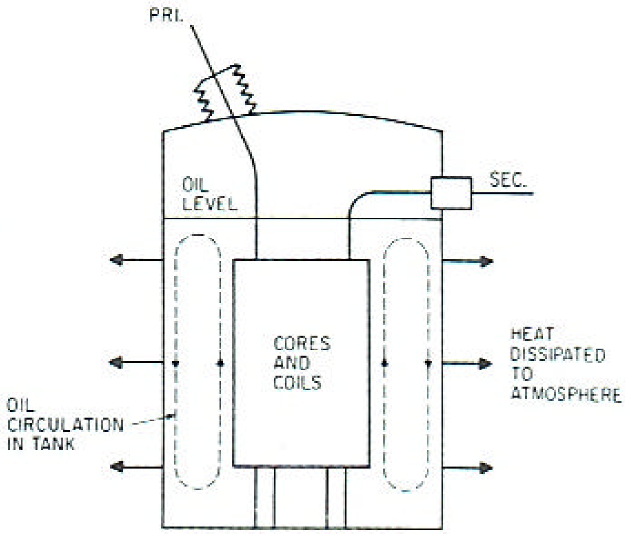

Figure 2. Cooling of Core-and-Coil Through Oil Convection

electrical energy is converted into heat energy, which is physically represented by the increase in temperature. The rate of heat generation in joules per cubic meter per second, represented by q_e, is given by the following expression [Bird, 2], [Schneider, 3] :

$$q_e = r_e I^2 \qquad (J/s - m^3 \ or \ W/m^3) \tag{1}$$

where r_e is the electrical resistivity of the winding, in $ohm - m$, and I is the current density, in A/m^2.

As for the core of the device, there are magnetic flux lines passing through the magnetic material. However, since the flux lines represent the alteration of the atomic structure of the core material due to external force (i.e., the magneto-motive force), and is not a measure of the flow of charged particles through the core material, therefore, there is no energy generation or loss within the core. The temperature of the core remains essentially unchanged once a steady state is reached.

Within the core-and-coil assembly, to be in a realistic situation, the variation of temperature in both the radial and the axial directions should be considered. By taking a differential geometrical element, the energy (i.e., the heat) flow in both the radial- and the axial-direction are analyzed and identified, then in conjunction with the fundamental law of heat conduction developed by Jean Fourier, the energy balance is taken. This results in the two-dimensional partial differential equation given below [Schneider, 3], [Carslaw, 5]:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{q_e}{k} = 0 \tag{2}$$

Where k is the thermal conductivity of the core-and-coil assembly, in W/m-C, T is the temperature, in ^oC, r is the radial coordinate, in m, and z is the axial coordinate, in m.

The solution of this partial differential equation involves two steps [Yeh, 1]. These are: (a) the introduction of a new dependent variable q, which is defined in Equation (3) below, and (b) the separation of q into two new dependent variables q_1 and q_2 , where q_1 is a function of the radial coordinate r only, i.e., $q_1(r)$, and q_2 is a function of the axial coordinate z only, i.e., $q_2(z)$. This relation is given by Equation (4) below:

$$q = T - T_2 - \frac{q_e}{2k} z(h-z) \quad or \quad T = q + T_2 + \frac{q_e}{2k} z(h-z) \tag{3}$$

In Equation (3), the term $q_e z(h-z)/2k$ represents the particular solution of the differential equation given in Equation (2), and T_2 is the temperature at the outer surface of the coil assembly. T_2 enters into the problem through the boundary conditions which will be introduced later.

$$q = q_1(r) q_2(z) \tag{4}$$

The final solution of q in Equation (4) can be written as follows [Yeh, 1]:

$$q = [C_1 \sin(mz) + C_2 \cos(mz)] \cdot [C_3 I_0(mr) + C_4 K_0(mr)] \tag{5}$$

Where m is a positive integer constant, which is called the separation constant. C_1, C_2, C_3 and C_4 are the integration constants, which are created during the process of solving the differential equation. $I_o(mr)$ is the zero order modified Bessel function of the first kind, and $K_o(mr)$ is the zero order modified Bessel function of the second kind [Yeh, 1], [Bronshtein, 4].

TWO MODELS OF THE CORE-AND-COIL ASSEMBLY

The core-and-coil assembly can be treated geometrically in two ways, namely:

Case (a): A cylindrical shell or tube, with a metallic (i.e., iron) core placed at the axial center. This is shown in Figure 3.

Case (b): A solid cylinder, with material spreading from the axis to the outer surface. This is shown in Figure 4.

Case (a): This model has been solved by the author in a previous publication [Yeh, 1]. The boundary conditions are specified in the following:

$$At\ r = r_1\ and\ 0 \le z \le h, the\ net\ heat\ transfer\ is\ zero$$

$$At\ r = r_2\ and\ 0 \le z \le h, the\ temperature\ is\ T = T_2,\ or\ q = q_2 = 0$$

$$At\ z = 0\ and\ r_1 \le r \le r_2, the\ temperature\ is\ T = T_2,\ or\ q = q_2 = 0$$

When these conditions are applied to Equation (5), the final solution is given below:

$$q = T - T_2 - \frac{q_e}{2k} \cdot z \cdot (h-z) = -\frac{4 q_e h^2}{k p^3} \sum_{n=0}^{\infty} \frac{I_1(g_n r_1) K_0(g_n r) + K_1(g_n r_1) I_0(g_n r)}{I_1(g_n r_1) K_0(g_n r_2) + K_1(g_n r_1) I_0(g_n r_2)} \cdot \frac{\sin(g_n z)}{(2n+1)^3} \tag{6}$$

$$where \quad g_n = p(2n+1)/h$$

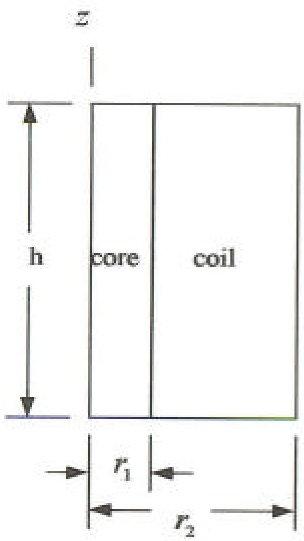

Figure 3. Core-and-Coil as a Cylindrical Tube

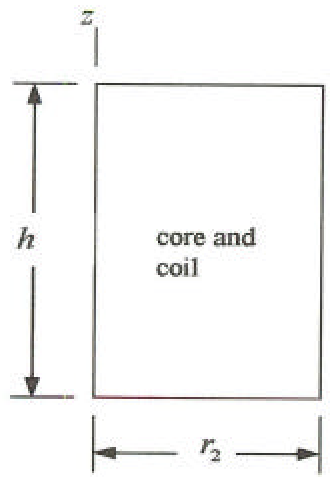

Figure 4. Core-and-Coil as a Solid Cylinder

In Equation (6), $I_1(x) \quad and \quad K_1(x)$ are the first-order modified Bessel function of the first kind and the first-order modified Bessel function of the second kind, respectively [Yeh, 1].

Case (b): The boundary conditions for this model are given below:

$$At\ r = 0\ and\ 0 \le z \le h, the\ temperature\ is\ \ finite$$

$$At\ r = r_2\ and\ 0 \le z \le h, the\ temperature\ is\ \ T = T_2\ , or\ q = q_2 = 0$$

$$At\ z = 0\ and\ \ 0 \le r \le r_2, the\ temperature\ is\ \ T = T_2\ , or\ q = q_2 = 0$$

$$At\ z = h\ and\ \ 0 \le r \le r_2, the\ temperature\ is\ \ T = T_2\ , or\ q = q_2 = 0$$

When these conditions are satisfied by Equation (5), the final solution becomes:

$$q = T - T_2 - \frac{q_e}{2k} \cdot z \cdot (h - z) = -\frac{4q_e h^2}{kp^3} \sum_{n=0}^{\infty} \frac{I_0(g_n r)}{I_0(g_n r_2)} \cdot \frac{\sin(g_n z)}{(2n+1)^3} \tag{7}$$

The mean profile temperature at any given axial location z , represented by $T_{a,z}$, is defined as follows:

$$pr_2^2 T_{a,z} = \int_0^{r_2} 2prTdr \tag{8}$$

This temperature has been determined and is given in the following:

$$T_{a,z} = T_2 + \frac{q_e}{2k} \cdot z \cdot (h - z) - \frac{8 q_e h^2}{k p^3 r_2} \sum_{n=0}^{\infty} \frac{I_1(g_n r_2)}{g_n I_0(g_n r_2)} \cdot \frac{\sin(g_n z)}{(2n+1)^3} \tag{9}$$

TEMPERATURE COMPUTATION

With the FORTRAN program for the numerical evaluation of the Bessel functions available from a previous research conducted by the author [Yeh, 1], a second FORTRAN program written for the computation of the temperature variation for Case (a), a third FORTRAN program has been prepared to obtain numerical results for Case (b). In these programs, the axial position, i.e., the z-coordinate, is divided into ten or more equal divisions, and for each axial position, the radial-coordinate, from the inner radius r_1 to the outer radius r_2, or from $r = 0$ to $r = r_2$, is divided into ten or more equal distances. The input data are listed below:

$$k = 200.00\,(W/m-C), \quad q_e = 4000{,}000.\,(W/m^3), \quad T_2 = 50.00\,(C), \quad r_1 = 0.054\,(m)$$

$$r_2 = 0.1800\,(m), \quad h = 0.3000\,(m)$$

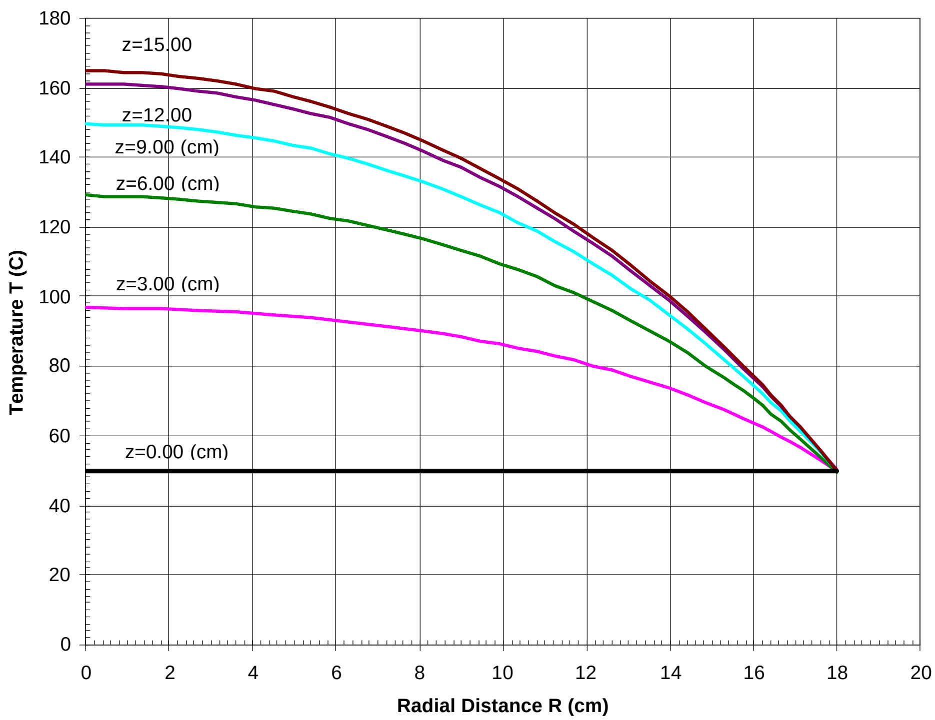

Figure 5. Temperature vs. Radial Coordinate, with Axial Coordinate as Parameter

Figure 5 is a graphical display of the temperature profiles for Case (b). The parameter used in the diagram is the axial coordinate z. As it is expected in actual operation of the transformer, the temperature decreases from the central axis of the core-and-coil assembly to its outer surface, where the heat is dissipated by the cooling fluid to the tank, and then carried away by the ambient air from the outside surface of the tank. Also, at a given radial location, the temperature is the highest at the mid-point of the axial direction, i.e., z =15.00 cm, and it decreases gradually to the top and bottom surfaces.

The mean profile temperature, represented by $T_{a,z}$, for Case (b) configuration is shown in Figure 6. Clearly, at the lower surface (i.e., z=0.00 cm) and the upper surface (i.e., z=30.00 cm), the mean temperature is also the surface temperature, where $T = T_2 =$ 50.00 C. At the middle surface of the height, where z=15.00 cm, the mean temperature is the highest, which is 110.42 C.

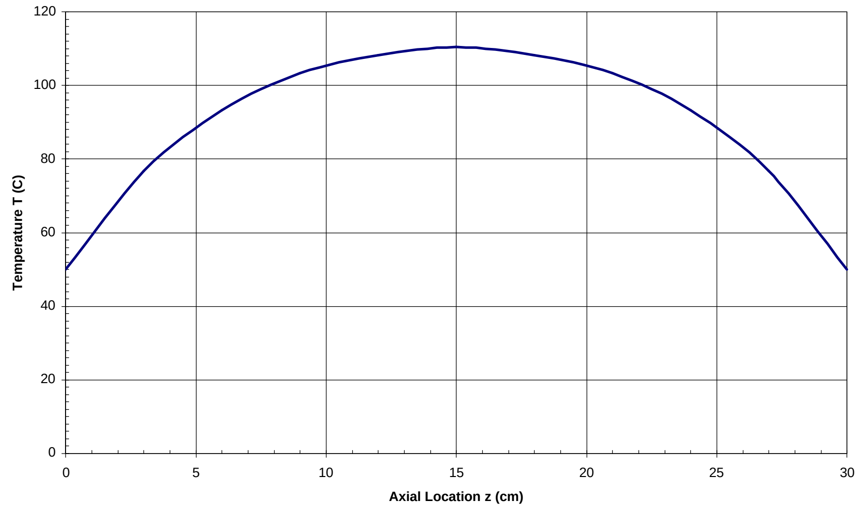

Figure 6. Mean Profile Temperature vs. Axial Location

In Figure 7, a comparison is made on the temperature variations between Case (a) and Case (b). It can be seen that at any location in the core-and-coil assembly, the temperature for the tubular coil is always lower then that for the

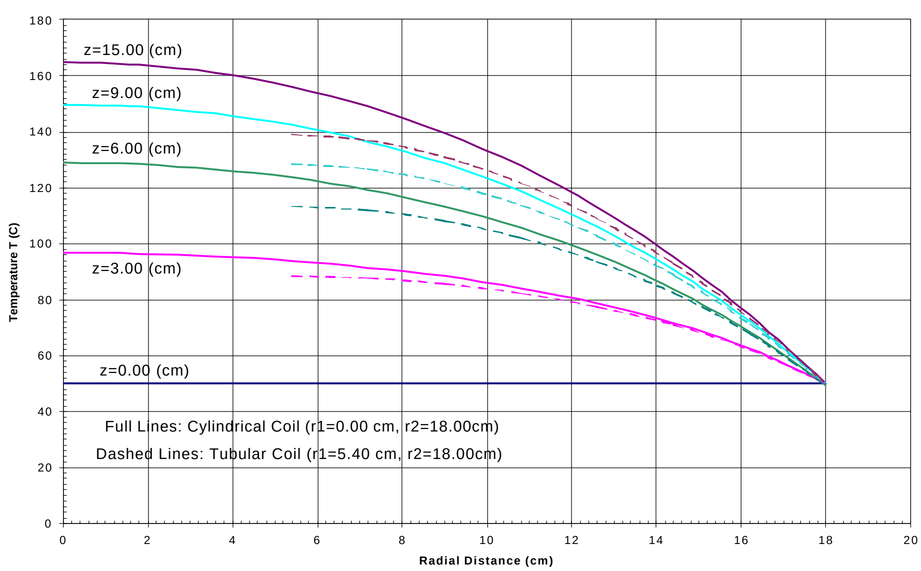

Figure 7. Comparison of Temperature Variation

cylindrical coil. For example, at r=5.40 cm, which corresponds to the inner surface of the tubular coil, and at the middle height of the cylinder, i.e., z=15.00 cm, the temperatures are 156.07 C versus 139.06 C, a difference of 17.01 C. At this axial location, for the solid cylinder core-and-coil, the temperature eventually increases to **164.92** C.

For the cylindrical tube model of the core-and-coil, it is of interest to study the effect of the variation of the inner radius, i.e., r_1, on the temperature variation of the entire assembly. In Figure 8, a small radius of $r_1 = 1.80$ cm is

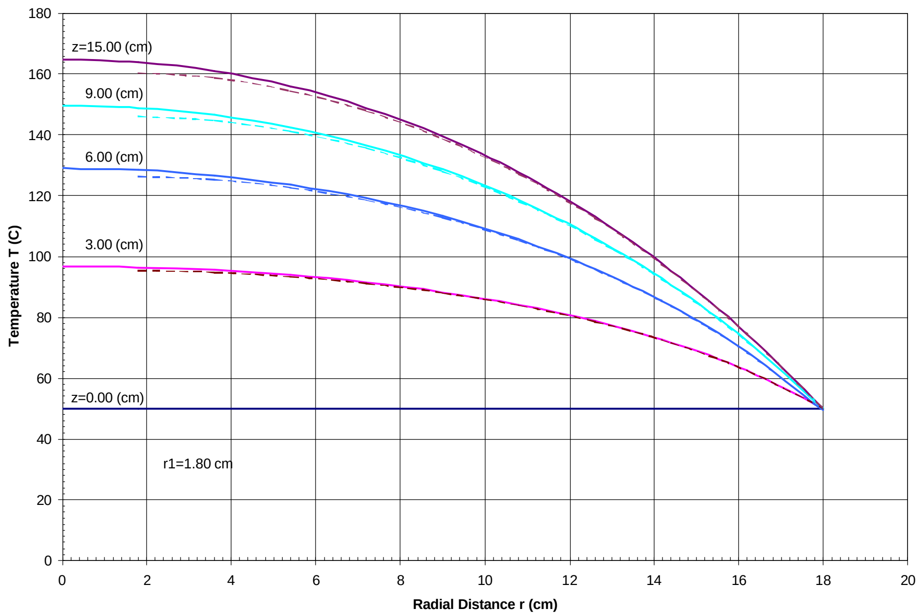

Figure 8. Comparison of Temperature Variation for a Small Value of r_1

used for the computation. It can be seen that between the cylindrical tube model and the solid cylinder model, at the same axial position (i.e., the same z-coordinate), the two temperatures are nearly the same, with a difference of no more than 1.00 C. Only when it is very close to the central axis of the cylinder, then the temperature difference becomes significant. For example, at z=15.00 cm and r=1.80 cm, the temperature is 163.95 C for the solid cylinder model, and 160.40 C for the cylindrical tube model, a difference of only 3.55 C.

From the mathematical point of view, it is clear that the solution for the solid cylinder model, i.e., Equation (7), is the asymptotic representation of the solution for the cylindrical tube model, i.e., Equation (6). However, Equation (6) can not be evaluated at $r_1 = 0$, since the values of the first order modified Bessel functions of the first kind and the second kind, $K_o(x)$ and $K_1(x)$, are not defined at $r_1 = 0$, both become positively infinite at the axis of the cylinder.

Figure 9 displays the isothermal lines in the coil structure for Case (b), i.e., the solid cylinder model. The boundary conditions specify the top, the bottom, and the outside surfaces to be all at a temperature of 50 ^oC. From these surfaces, heat is carried away by the cooling oil through natural convection. The temperature is the highest at the axial center point of the cylinder, where the axial coordinate is z=15.0 cm, and the radial coordinate is r=0.0 cm. The maximum temperature is T=164.92 ^oC. From this point, the temperature decreases both in the axial and radial directions.

For the purpose of comparison, the isothermal lines in the coil for Case (a), i.e., the cylindrical tube core-and-coil model, is shown in Figure 10. The mathematical representations for the temperature variation are different for the two cases, i.e., Case (a) contains the summation of modified Bessel functions of the first kind of zero order and first order, modified Bessel functions of the second kind of zero order and first-order, and the sine function. While Case (b) contains the summation of modified Bessel functions of the first kind of zero order and first order, and the sine function. However, the general trends of the temperature variation are very similar. At any given point in the core-and-coil, the temperature as represented by the solid cylinder model is always higher then that given by the cylindrical tube model.

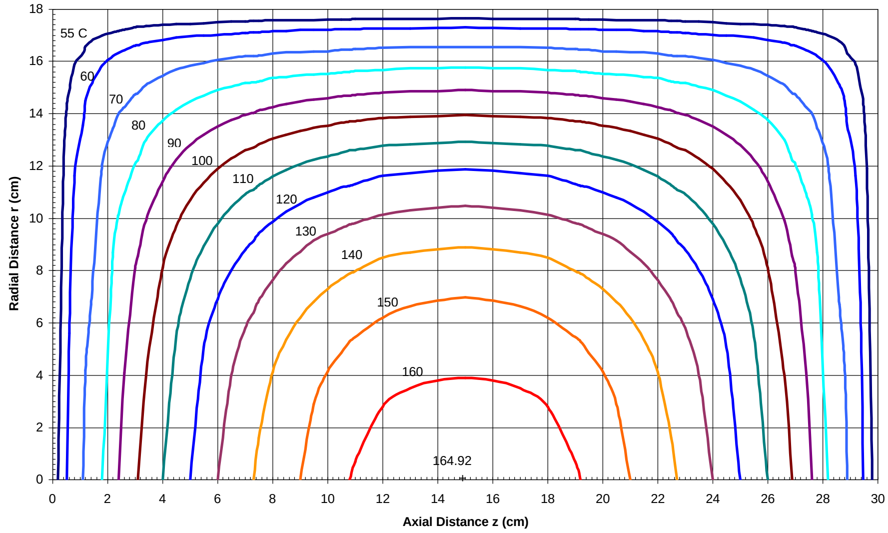

Figure 9. Isothermal Lines in the Coil Structure for Case (b)

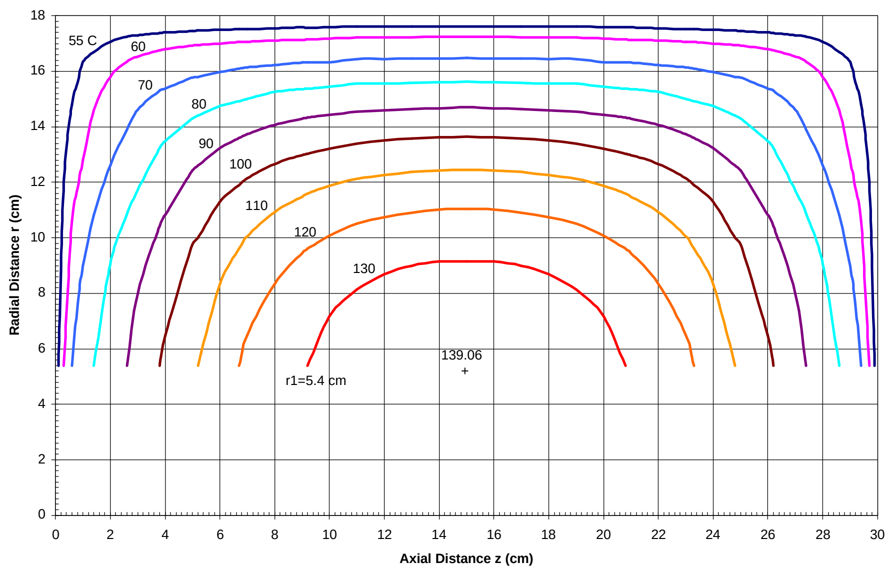

Figure 10. Isothermal Lines in the Coil Structure for Case (a)

SUMMARY

On the study of the temperature variation in the core and coil of a distribution transformer, the two-dimensional analysis indicates that the mathematical solution can be expressed in terms of the product of infinite series and transcendental functions. The infinite series are the modified Bessel functions, and the transcendental functions are the trigonometric functions. Using computer programs developed in the present study, the numerical results can be obtained by the summation of successive terms in the solution, until a term is reached where the numerical value is very small, such as 1×10^{-6}, so that any following terms can be dropped. Within the cylindrical shell of the coil assembly, the numerical results show that the temperature is the highest at the axial center point of the inner surface for the cylindrical tube model, or at the geometrical center for the solid cylinder model. From this location, the temperature decreases both in the axial and radial directions. The predicted temperature at any point based on the solid cylinder model is always higher than that for the cylindrical tube model.

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