

Maple as an Instructional Tool

Terence A. Weigel¹

Abstract – Computer algebra software such as Maple is an important component of the engineer’s toolkit, much as are Matlab, MathCAD and Excel. However, Maple can also be used effectively as an instructional tool, similar to the way in which Powerpoint and Adobe Acrobat are used. At the University of Louisville, civil engineering students specializing in the structures area are required to take CEE 621, *Finite Element Analysis for Structural Engineers*. Finite element methodology is heavily invested with calculus, linear algebra, differential equations and numerical methods. Matrix manipulation, differentiation, definite and indefinite integration, integration by parts, Taylor series, Green’s Theorem, solution of partial differential equations, numerical integration and solution of large systems of simultaneous equations are all essential parts of this course. At U of L, Maple is used as an instructional tool in CEE 621. As such it is used to: present complex derivations, work example problems, investigate “what if” scenarios and answer student questions

Keywords: Maple, instruction, calculus, equations, matrix.

INTRODUCTION

At the University of Louisville, civil engineering students specializing in the structures area are required to take CEE 621, *Finite Element Analysis for Structural Engineers*. Finite element theory and application are heavily invested with calculus, linear algebra, differential equations and numerical methods. Matrix manipulation, differentiation, definite and indefinite integration, integration by parts, Taylor series, Green’s Theorem, solution of partial differential equations, numerical integration and solution of large systems of simultaneous equations are all essential parts of this course.

Traditional presentation of finite element theory would involve class time to derive manually important equations and demonstrate important principles, solve differential equations, invert matrices and solve systems of equations. Done in this manner, the presentations are tedious and consume significant amounts of precious class time. In CEE 621, the author takes a different approach. Almost all of the lectures have been prepared in Maple [Maplesoft, 1] - all of the necessary calculus and matrix manipulation are accomplished using the capabilities of Maple. This approach to presenting CEE 621 material offers numerous advantages over the traditional approach:

- Derivations and example problems can be presented in less time
- “What if” scenarios can be investigated “on the fly” in the classroom
- Lectures can be posted on the Internet. Students can download the lectures and examine them in detail. Modifications, corrections and additions to lectures can be made easily and reposted on the Internet quickly
- Maple files can be provided students, allowing them to do their own “what if” investigation
- Maple allows many graphical presentations that would be difficult with a traditional approach
- Maple features can be used to respond immediately to questions and possibly with more complete responses

¹ Department of Civil and Environmental Engineering, University of Louisville, Louisville, KY 40292, taw@louisville.edu

EXAMPLE LECTURE IN MAPLE

Figure 1 shows the CEE 621 lecture on isoparametric elements. Items shown in light blue are Maple output. Items shown in black are produced using the text input feature of Maple. For reasons explained in the next section, Maple input commands are not included in Figure 1.

Introduction to Isoparametric Elements

The rectangular elements we have seen previously are easy to formulate but are impractical in the sense that they cannot be used easily in meshes where the geometry is complicated. With rectangular elements, complicated geometry can be represented accurately only with a very fine mesh. We now introduce parametric elements. Initially rectangular parametric elements can be distorted so that irregular geometry of a body can more easily be represented. With the isoparametric formulation, quadrilateral elements can be distorted into non-rectangular shapes, and, in fact, can be distorted to produce elements with curved sides.

A basic operation in finite element analysis is the transformation from nodal displacements to displacements within an element. We have used this operation in formulating all of the elements we have studied to date. Recall that this operation had the following form:

$$\{u\} = [N]\{d\}$$

where, in three dimensions:

$\{u\} = [u \quad v \quad w]^T = [N]\{d\}$ and are the x , y and z components of displacement at points inside the element, respectively;

$[N]$ is the matrix of shape functions, and;

$\{d\}$ is the matrix of x , y and z components of displacement at the element nodes.

We now introduce a second transformation, one which handles element geometry. Specifically:

$\{x\} = [x \quad y \quad z]^T = [\tilde{N}]\{c\}$ and are the x , y and z are the coordinates of a physical point in space;

$[\tilde{N}]$ is the appropriate geometry transformation matrix, and;

$\{c\}$ is the matrix of the physical coordinates at the element nodes.

Parametric elements are formulated in what are called *natural*, *reference* or *auxiliary* coordinates. We will use ξ , η and ζ to represent the natural coordinate system. Both $[N]$ and $[\tilde{N}]$ are functions of the natural coordinates. If $[N]$ is identical to $[\tilde{N}]$, the formulation is called *isoparametric*.

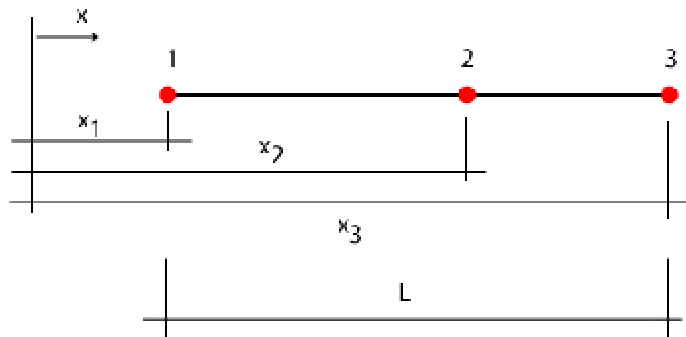
Figure 1 - Isoparametric Element Lecture in Maple

Example of an Isoparametric Bar Element

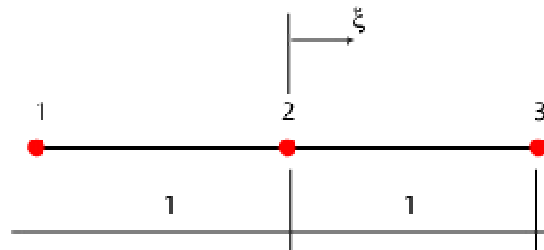
The following example does not illustrate the full power of the isoparametric formulation, but does serve to show the process of element formulation in simple terms.

Consider a three-noded bar. In the upper portion of the figure, the bar is shown in physical coordinates (x) and in the lower part of the figure, the bar is shown in natural coordinates (ξ). Note that the element has an off-center internal node (Node 2).

We formulate the geometry mapping in terms of the natural coordinate ξ . The external nodes lie at $\xi = \pm 1$. In the natural coordinate system, the element always has length of 2, and Node 2 lies at mid-length, regardless of the geometry and nodal layout of the element in physical coordinates. Our formulation is isoparametric because we use the same interpolation for both geometry and for displacement.



Physical element



Mapped element

Our formulation will require six constants a , three to map element displacements and three to transform geometry. Therefore we define for the mapping from natural to physical coordinates:

Figure 1 (continued) - Isoparametric Element Lecture in Maple

$$\begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$x = a_1 + \xi a_2 + \xi^2 a_3$$

The values a_1, a_2, a_3 are unknown constants that are yet to be determined.

We use a similar transformation to get element displacements from nodal displacements:

$$a_{displacement} := \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix}, \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

$$u = a_4 + \xi a_5 + \xi^2 a_6$$

To obtain the shape function matrix (which is also used to transform physical coordinates), proceed as we have before. We write the equation for x using the value of the natural coordinates at each node. This yields:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Combining this expression with the expression $x = a_1 + a_2\mathbf{x} + a_3\mathbf{x}^2$ yields:

$$x_{nodes} := \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

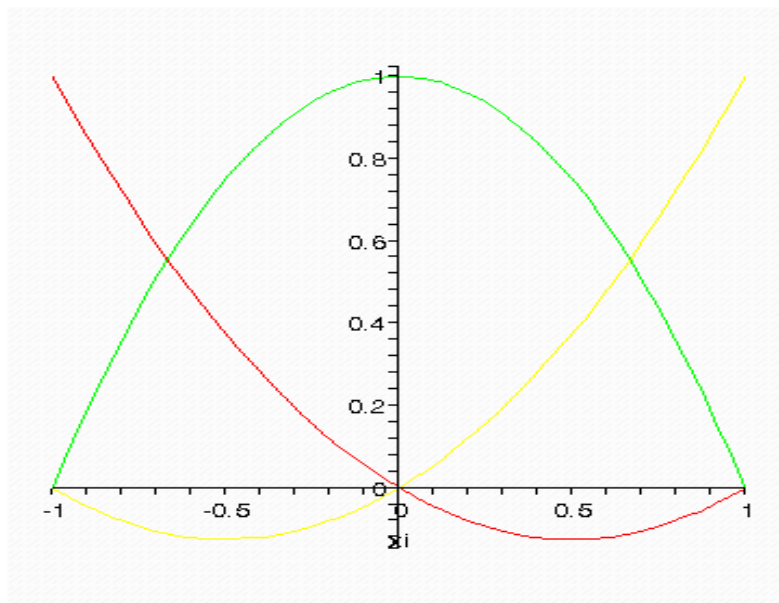
Figure 1 (continued) - Isoparametric Element Lecture in Maple

$$\begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

The product of the first two matrices in this triple product is the shape function matrix, $[N]$.

$$N := \begin{bmatrix} -\frac{1}{2}\xi + \frac{1}{2}\xi^2 & 1 - \xi^2 & \frac{1}{2}\xi + \frac{1}{2}\xi^2 \end{bmatrix}$$

Plots of the shape functions are shown in the figure below.



The axial strain in the element is:

$$e_x = \frac{d}{dx} u(x) = \frac{\partial}{\partial x} N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

To differentiation $[N]$ with respect to x , we must invoke the *chain rule*, because $[N]$ is expressed as a function of ξ . Because $\frac{dx}{d\xi}$ is not available, we must first compute its inverse, $\frac{d\xi}{dx}$.

Figure 1 (continued) - Isoparametric Element Lecture in Maple

$$e_x = \left(\frac{\partial}{\partial x} \xi, \frac{\partial}{\partial \xi} N, u_{nodes} \right)$$

The chain rule must be used because \mathbf{N} is expressed in terms of \mathbf{x} rather than x . However, $\frac{d\mathbf{x}}{dx}$ must first be determined. It is obtained by finding $\frac{dx}{d\mathbf{x}}$. Let the symbol J denote this inverse.

$$J = \frac{\partial}{\partial \xi} x$$

Continuing we have:

$$J = \frac{\partial}{\partial \xi} N \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

And:

$$J = \left[\left(-\frac{1}{2} + \xi \right) x_1 - 2\xi x_2 + \left(\frac{1}{2} + \xi \right) x_3 \right]$$

J is called the Jacobian. It represents a scale factor that relates physical length dx to reference length $d\xi$

Axial strain in the element may now be expressed as:

$$e_x = \frac{\frac{\partial}{\partial \xi} u}{J}$$

$$e_x = B \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

where the matrix relating strain to nodal displacements is:

$$B := \frac{1}{J} \begin{bmatrix} -\frac{1}{2} + \xi & -2\xi & \frac{1}{2} + \xi \end{bmatrix}$$

Figure 1 (continued) - Isoparametric Element Lecture in Maple

Finally, the stiffness matrix $[k]$ is given by (in physical coordinates):

$$k = \int_0^L B^T E B A dx$$

It is generally more convenient to evaluate $[k]$ in natural coordinates thus:

$$k = \int_{-1}^1 B^T E B A J d\xi$$

Figure 1 (continued) - Isoparametric Element Lecture in Maple

Maple Input

Maple input commands are not shown in Figure 1. This is done to satisfy space limitations of this paper, but also because Figure 1 shows exactly how the lecture is presented to students. Suppressing Maple input gives the lecture a “cleaner” look and results in a document that is less confusing for the reader. The user of the Maple file can toggle between hiding or revealing input lines. A few lines of Maple input / output is shown in Figure 2. In Figure 2, Maple input is shown in red, and Maple output is shown in light blue.

The mapping from natural coordinates to physical coordinates is given by:

> x[nodes]:=Vector(3,[x1,x2,x3]);

$$x_{nodes} := \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

$$x_{geometry} = \begin{bmatrix} a_1 + \xi a_2 + \xi^2 a_3 \end{bmatrix}$$

> natural, a[geometry];

$$\begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Figure 2 - Maple Input

```
> x=(natural . a[geometry])[1];
```

$$x = a_1 + \xi a_2 + \xi^2 a_3$$

The values a_1, a_2, a_3 are unknown constants that are yet to be determined.

We use a similar transformation to get element displacements from nodal displacements:

```
> natural . a[displacement];
```

$$\begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix}, \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

```
> u=(natural . a[displacement])[1];
```

$$u = a_4 + \xi a_5 + \xi^2 a_6$$

To obtain the shape function matrix (which is also used to transform physical coordinates), proceed as we have before. We write the equation for x using the value of the natural coordinates at each node. This yields:

```
> Nodes:=Matrix(3,3,[[1,-1,1],[1,0,0],[1,1,1]]);
```

```
> Nodes,a[geometry];
```

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Combining this expression with the expression $x = a_1 + a_2\mathbf{x} + a_3\mathbf{x}^2$ yields:

```
> x[nodes]:=Vector(3,[xi[1],xi[2],xi[3]]);
```

$$x_{nodes} := \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

Figure 2 - Maple Input

```
> with(linalg,inverse):
```

```
> InverseNodes:=inverse(Nodes):
```

```
> natural,evalm(InverseNodes),x[nodes];
```

$$\begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

Figure 2 - Maple Input

Upon examining Figures 1 and 2, it is immediately obvious that using Maple accomplishes two things at the expense of doing only one. Creating Maple input generates a correctly formatted mathematical expression, while at the same time performing the implied operation. Normally, lectures created in electronic format require use of a special “equation editor” if they are to be shown in standard mathematical format similar to that found in a textbook. Maple implicitly furnishes the equation editor feature, while at the same time performing the mathematical operation.

OTHER USES

By its nature, finite element analysis is computationally intensive. Only the simplest problems can be worked by hand. Solving even small realistic problems requires software assistance. Of course there are many commercial software packages available to assist the finite element analyst, and these programs have a role to play even in an introductory course. Unfortunately, exclusive reliance on commercial software robs the student of the opportunity to experience the complete process of solving finite element problems. Here Maple also offers help. Maple may be used by students to do the “grunt work” of matrix manipulation, but he / she is still required to create and direct the Maple operations.

SUMMARY AND CONCLUSIONS

This paper presents a methodology by which Maple software may be used as an instructional tool. Presenting lectures using Maple (or similar symbolic manipulation software) saves time and yields an interactive and flexible environment for learning. Specifically Maple offers the following features when used as an instructional tool:

- Derivations and example problems can be presented in less time
- “What if” scenarios can be investigated “on the fly” in the classroom

- Lectures can be posted on the Internet. Students can download the lectures and examine them in detail. Modifications, corrections and additions to lectures can be made easily and reposted on the Internet
- Maple files can be provided students, allowing them to do their own “what if” investigation
- Maple allows many graphical presentations that would be difficult with a traditional approach
- Maple features can be used to respond immediately to questions and possible with more complete responses

In addition, Maple can be used to work example problems and can be used by students to complete assigned homework.

REFERENCES

- [1] Maple 9.00, Maplesoft, Waterloo Maple Inc., 2003.

Terence A. Weigel

Terry Weigel holds a PhD from the University of Kentucky and has taught course related to structural engineering and computer applications at the University of Louisville for 27 years. He is currently developing an online system for grading homework and similar assignments in courses such as statics, structural analysis and steel design. His research interest is behavior and design of masonry structures, particularly as related to seismic loading. He is a member of ASEE, ASCE, ACI, EERI, SSA, and TMS.