The Use of Mathcad in a Convective Heat Transfer Course

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Abstract

Experiences using Mathcad in a graduate-level convective heat transfer course are described. Many of the classical equations of laminar free or forced convective heat transfer are second- or third-order nonlinear ordinary differential equations, or systems of such equations, that are boundary-value problems. Mathcad functions make numerical solution of these classical equations relatively simple and quick--thus permitting routine solutions either in class or as homework. The symbolic manipulation capability of Mathcad is exploited for integral solution techniques. Additionally, the utility of using Mathcad to examine turbulent flow relations (law of the wall and power-law profiles for instance) is demonstrated. Examples of Mathcad applications are given, and the problems encountered are discussed. The use of Mathcad was judged to enhance the presentation of course material and to provide a more complete understanding of convective heat transfer solution techniques.

Background

A convective heat transfer course is often a part of graduate education in mechanical engineering, especially for students focusing on the thermal sciences. The thrust of the coverage of convective heat transfer in such courses is towards fundamentals and understanding, rather than intense numerical procedures.

Most graduate students in a first graduate course in convective heat transfer have taken only a typical undergraduate course in heat transfer. At the undergraduate level, most heat transfer textbooks [for example, Incropera and deWitt (1996), Mills (1999), and Holman (2002)] present solutions with little or no detail as to the computational procedures employed, although increasingly general undergraduate heat transfer textbooks come with some sort of computer-based enhancements. Graduate level convective heat transfer textbooks [Bejan (1995), Kays and Crawford (1996), Burmeister (1993), and Oosthuizen and Naylor (1999)] generally provide more details of solution procedures and computer software resources, but applications overhead can be high for a single course.

Graduate-level first courses in convective heat transfer typically follow the pattern: derivation of the Navier-Stokes equations, boundary-layer theory, solutions of classic laminar boundary-layer flows and fully-developed laminar duct flows, transition considerations, and turbulent flow fundamentals and applications. Emphasis at the graduate level is placed on procedures to solve many of the equations and usually several assignments involving computer program development and/or software exercises are made. Often, solution techniques for many of the classic convective heat transfer equations are discussed and then the numerical results are presented, with the student generally accepting that the solutions are arithmetically intense to obtain.

User-friendly software systems such as Mathcad are now available. These software systems permit the easy, routine numerical solution of many of the equations of convective heat transfer. Even in cases where difficulties are experienced when using systems such as Mathcad, students are able to fathom and appreciate the difficulties. This paper will explore the use of Mathcad to enhance understanding and acceptance of results in a graduate convective heat transfer course. An important advantage of using Mathcad is that homework problems here-to-fore judged as requiring more coding/debugging efforts than benefits become feasible assignments. Examples of the use of Mathcad in a

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convective heat transfer course will be presented. The difficulties for Mathcad presented by the examples will also be explored.

Examples

Many of the classical differential equations of convective heat transfer are boundary-value problems associated with higher-order, non-linear differential equations or systems of such equations. Within the framework of ordinary differential equations, boundary-value problems have boundary conditions specified at more than one value of the independent variable and are thus distinct from initial-value problems, which have all boundary conditions specified at a single value of the independent variable. Initial-value problems can be solved by numerical integration from the location of the specification of the boundary conditions. Boundary-value problems generally require some sort of iterative procedure involving satisfaction of the boundary conditions when specified at different values of the independent variable.

Laminar Boundary-Layer Flow

The first example considered is for laminar, flat-plate boundary-layer flow of a constant property fluid. The two-dimensional, laminar, constant property boundary-layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 u}{\partial y^2}$$

with the boundary conditions: $y = 0$, $u(0) = v(0) = 0$ and $T = T_s$ and $y \to \infty$, $u = U_\infty$ and $T = T_\infty$. Extensive details and discussion on both the derivation and solution of these equations are provided in Bejan (1995), Kays and Crawford (1996), Burmeister (1993), and Oosthuizen and Naylor (1999).

Integral Method

The integral solution procedure starts with specifications of the functional forms of the velocity and temperature profiles. Following the nomenclature of Bejan (1995), the functional forms of he velocity and temperature profiles are

$$\frac{u}{U_\infty} = m(n) \qquad n \le 1 \qquad \text{and} \qquad \frac{T - T_o}{T_\infty - T_0} = m(p) \qquad n \le 1 \tag{2}$$
$$= 1 \qquad n \succ 1 \qquad\qquad\qquad = 1 \qquad n \succ 1$$

where $n = \dfrac{y}{\delta}$ and $n = p \cdot del$ with $del = \dfrac{\delta_T}{\delta}$. The integral solution results [Bejan (1995)] are as follows:

$$\frac{\delta}{x} = a_1 \cdot \text{Re}_x^{-0.5} \qquad C_f = a_2 \cdot \text{Re}_x^{-0.5} \qquad \delta_T = del \cdot a_1 \cdot x \cdot \text{Re}_x^{-0.5} \qquad Nu_x = \frac{\text{Re}_x^{0.5}}{del \cdot a_1}\left(\frac{dm}{dp}\right)_0 \tag{3}$$

where $\quad a_1 = \left[\dfrac{2\cdot(dm/dx)_0}{\displaystyle\int_0^1 m(n)\cdot[1-m(n)]\cdot dn}\right]^{\frac{1}{2}}$ and $a_2 = \left[2\cdot(dm/dx)_0 \cdot \displaystyle\int_0^1 m(n)\cdot[1-m(n)]\cdot dn\right]^{\frac{1}{2}} \tag{4}$

And the energy integral equation for high Pr fluids ($\text{Pr} > 1$) yields

$$\text{Pr} = \frac{2 \cdot (dm/dx)_0}{(del \cdot a_1)^2 \cdot \int_0^1 m(del \cdot p) \cdot [1 - m(p)] \cdot dp} \tag{5}$$

which provides the solution of del as a function of Pr. The procedure is to specify a functional form for $m(n)$ and to carry out the indicated integrations. Linear, quadratic, cubic, quartic, and sine profiles are among those commonly used. Mathcad easily accomplishes the evaluations of the above expressions for any typical profile. For example, the cubic profile function is

$$m(n) = \frac{n}{2} \cdot (3 - n^2) \tag{6}$$

Figure 1 presents the Mathcad worksheet for the solution with the cubic profile. The profile is defined, and the required expressions ($dmdn(0)$, a_1, and a_2) evaluated. The energy integral relation yields an expression for Pr as a function of del. The symbolic capability of Mathcad is utilized to evaluate the integral and to extract the only positive root, and the constants are evaluated. The final result is $Nu_x = 0.331 \cdot \text{Re}_x^{0.5} \cdot \text{Pr}^{0.333}$. A number of classroom exercises or homework problems based on the integral Mathcad procedure is possible.

Similarity Solution:

By introducing the similarity variable, $\eta = y \sqrt{\dfrac{U_\infty}{2 \cdot v \cdot x}}$, with $u = U_\infty \cdot f'(\eta)$ and $\theta(\eta) = \dfrac{T - T_0}{T_\infty - T_0}$, Equations (1) reduce to

$$\begin{aligned} f \cdot f'' + f''' &= 0 \\ \theta'' + \text{Pr} \cdot f \cdot \theta' &= 0 \end{aligned} \tag{7}$$

with the boundary conditions: $\eta = 0$, $f(0) = f'(0) = \theta(0) = 0$ and $\eta \to \infty$, $f'(\infty) = \theta(\infty) = 1$. Equations (7) constitute a system of non-linear ordinary differential equations that are boundary-value problems. Perhaps the most simplistic approach to solving boundary-value problems is the shooting technique. The procedure involves iteratively finding values of missing boundary conditions at $\eta = 0$ such that the boundary conditions as $\eta \to \infty$ are satisfied. The technique is usually called shooting and is indeed reminiscent of adjusting the velocity, elevation, and azimuth of an artillery piece to provide projectile impact at a desired location. The laminar boundary-layer flow solution thus requires finding $f''(0)$ and $\theta'(0)$ such that $f'(\infty)$ and $\theta(\infty) = 1$.

The Mathcad solution to this problem involves using function **SBVAL** to find the values of $f''(0)$ and $\theta'(0)$ and then using a conventional higher-order Runge-Kutta function, such as **RKFIXED** in Mathcad, to integrate the equations as an initial-value problem. The simplest way to discuss the Mathcad procedure is to use the annotated Mathcad worksheet that is reproduced as Figures 2 and 3. As will be demonstrated later in this paper, **SBVAL** can be used to solve for more than one initial condition, but this example, the momentum equation ($f \cdot f'' + f''' = 0$) can be solved independently of the energy equation, and the energy equation then solved. As in any iterative procedure, an initial guess must be made and is specified as $\mathbf{v_0}$. The third-order differential equation is cast in the usual fashion as three, first-order differential equations in $\mathbf{d(x,y)}$. Both known [$f(0) = f'(0) = 0$] and guessed [$f''(0)$]initial values are used in the vector array **IC**. The convergence criterion is specified in **test** with idea being to drive the value to zero by refining the guessed values of $f''(0)$. **SBVAL** is then called with the output containing the converged value of $f''(0)$. The value shown as **s** is the required initial condition that results in $f'(\infty) = 1$.

With $f(0)$, $f'(0)$, and $f''(0)$ known, the differential equation can be integrated as an initial value problem to obtain the solution. The function $\mathbf{d(x,y)}$ for the final Runge-Kutta integration contains five (5) rather than the three (3)

differential equations used in the **SBVAL** setup. The additional equations are added to provide the integrations needed for the displacement and momentum thicknesses. The solution, which is available in the f, fp, and fpp vectors, is illustrated in the plot presented in the figure.

With the solution to the momentum equation found, the energy equation can be solved. **SBVAL** can be used to simultaneously find the missing initial conditions for both the momentum and energy equations, but for large (>>1) and small (<<1) Pr numbers, **SBVAL** does not possess sufficient robustness to easily solve the simultaneous problem. However, with the missing initial condition for the momentum equation, the energy equation can be solved using the **SBVAL**-based procedure. Figure 3 provides an example of the Mathcad procedure for solution of the energy equation for arbitrary Pr number. The example presented is for Pr = 100. The missing initial condition, $\theta'(0)$, in the energy equation is found using **SBVAL**. As with the momentum equation, the missing initial condition is guessed (to start the iterative process), the vector of initial conditions defined, the momentum and energy equations specified in **d(x,y)**, and the **test** function defined. **SBVAL** is invoked and returns $\theta'(0)$, the missing initial condition. Once $\theta'(0)$ is known, the solutions to both the momentum and energy equations are found using **RKFIXED**. The solution results are parsed from the z array and are available for use in a problem or for plotting.

The procedures delineated above will be followed in all of the boundary-value examples. Mathcad provides a rapid, easily-implemented protocol for many of the classical equations encountered in an laminar convective heat transfer. Moreover, the nature of a boundary-value problem is reinforced by the procedure with the explicit steps used for the **SBVAL** call followed by the **RKFIXED** integration.

Fully-Developed Laminar Duct Flows

Fully-developed laminar duct flows are fundamental considerations in a convective heat transfer course. For such flows, two conditions are usually examined: uniform heat flux and uniform wall temperature. For all fully-developed scenarios examined in this paper, the velocity profile is invariant and is given by the Poiseuille parabolic profile

$$u(r) = 2 \cdot U \cdot \left[1 - \left(\frac{r}{r_0} \right)^2 \right]$$
(8)

where U is the average velocity. Each of the two conditions will be considered.

Uniform Heat Flux

For uniform heat flux in fully-developed flow in a circular tube, the energy equation reduces to

$$\theta'' + \frac{\theta'}{r^*} + 2 \cdot \left[1 - \left(r^* \right)^2 \right] = 0$$
(9)

subject to the boundary conditions $\theta'(1) = 0$ and $\theta(0) = 0$. The form of Equation (9) is such that $Nu = \frac{1}{\theta_m}$ where $\theta_m = 4 \int_0^1 \left[1 - \left(r^* \right)^2 \right] \cdot \theta(r^*) \cdot r^* \cdot dr^*$. The desired results of the solution procedures are the Nu and the temperature profile. Although the system defined by Equation (9) can be solved in closed form, the case is a good one to explain the numerical technique using Mathcad. Consider Figure 4, the Mathcad worksheet illustrating the procedure. As in the previous example, the problem is a boundary-value problem and **SBVAL** will be used. The missing initial condition is $\theta(0)$. **SBVAL** returns $\theta(0) = 0.375$. **RKFIXED** is then invoked to integrate Equation (9). θ_m can be expressed as a differential equation by using the integrand of the θ_m definition. Hence, the last entry in **d(r, θ)** is a differential equation for $\theta_m(r^*)$; the last value is the desired result for θ_m. The initial conditions are defined, and **RKFIXED** invoked. The individual values are parsed from the **z** array. Using the last value from θ_m, the value of the Nu number for the constant heat flux, fully-developed solution is 4.3656.

Constant Surface Temperature

The constant surface temperature, fully-developed duct flow solution is similar to the constant heat flux solution, but an important difference between the two does exist. Consider the formulation for the constant surface temperature:

$$\phi'' + \frac{\phi'}{r^*} + 2 \cdot Nu \cdot \left[1 - \left(r^* \right)^2 \right] \cdot \phi = 0 \tag{10}$$

subject to the boundary conditions $\phi'(0) = 0$ and $\phi(1) = 0$. At the surface ($r^* = 1$) the heat flux must satisfy $\phi'(1) = Nu/2$. The desired results of the solution procedures are the Nu and the temperature profile. The constant surface temperature solution differential equation contains the Nu number as a factor, and the solution also demands $\phi'(0) = Nu/2$. One way to work the problem is to guess a value of Nu, integrate the differential equation with $\phi'(0) = Nu/2$, and verify that $\phi'(1) = 0$. The constant surface temperature Nu number is 3.6568.

In addition to Nu number differences, the constant surface temperature and the constant heat flux temperature profiles are also different. Figure 5 contains plots of both temperature profiles. The constant surface temperature profile is fuller near the centerline ($y^* = 1$) than the constant heat flux. As indicated by the relative closeness of the Nu numbers (4.36 versus 3.66), the surface temperature gradients are quite close.

Laminar Free Convection on a Vertical Wall

For laminar free convection on a vertical wall, the system of differential equations appears as (with y the vertical and x the horizontal directions):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \nu \frac{\partial^2 v}{\partial x^2} + g \cdot \beta \cdot (T - T_\infty) \tag{11}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions : $x = 0$, $u(0) = v(0) = 0$ and $T = T_0$ and $x \to \infty$, $v = 0$ and $T = T_\infty$. The similarity variable is $\eta = \frac{x}{y} Ra_y^{\frac{1}{4}}$ where $Ra_y = \frac{g \cdot \beta \cdot \Delta T \cdot y^3}{\alpha \cdot \nu}$, and the stream function and temperature profile are defined as

$$\Psi = \alpha \cdot Ra_y^{\frac{1}{4}} \cdot F(\eta, \Pr) \qquad \text{and} \qquad \theta(\eta, \Pr) = \frac{T - T_\infty}{T_0 - T_\infty} \tag{12}$$

With the aforementioned, Equations (11) reduce to

$$\frac{3}{4} \cdot F \cdot \theta' - \theta'' = 0$$

$$F''' - \theta + \frac{1}{\Pr} \cdot \left(\frac{1}{2} \cdot F' \cdot F' - \frac{3}{4} \cdot F \cdot F'' \right) = 0 \tag{13}$$

with the boundary conditions: $\eta = 0$, $F(0) = F'(0) = 0$ and $\theta(0) = 1$ and $\eta \to \infty$, $F'(\infty) = \theta(\infty) = 1$. Equations (13), in common with the previous examples, represents a boundary-value problem. However, for laminar free convection, the two equations are coupled via the temperature (θ) and must solved simultaneously. The procedure is the same as for the previous boundary-value problems except that two initial conditions must be found. Figure 6 contains the Mathcad worksheet for the solution of the coupled differential equations for Pr = 10. Guesses are made for the missing initial conditions, **SBVAL** is invoked, and **RKFIXED** is used to integrate the system of ordinary differential equations. However, care must be taken to ensure that the maximum value of η is sufficient to capture the extent of the

viscous flow field. **SBVAL** will attempt a solution for any value of η_{max} and will force the solution to be contained within the specified value. Thus, the tactic is to select an η_{max} with initial guessed values of the missing initial conditions that yield a solution. Then η_{max} is increased using the converged value from SBVAL. This procedure is continued until the behavior is asymptotic with η_{max}.

The solution for Pr = 10, presented in Figure 6, was generated using the procedure delineated in the previous paragraph. The temperature profile is asymptotic at $\eta \approx 5$, but the velocity profiles extends to $\eta \approx 18$! Thus, mass flow is entrained in the freestream region where the temperature is T_∞. Indeed, for Pr = 10 more than 50 percent of the entrained mass is at T_∞.

Boundary-Layer Flow with Viscous Dissipation

Viscous dissipation is important in high-speed flows or in high Pr number flows. When viscous dissipation effects are important, the energy equation, the last of Equations (1), becomes

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 u}{\partial y^2} + \frac{\alpha \cdot \mathrm{Pr}}{c_p}\cdot\left(\frac{\partial u}{\partial y}\right)^2 \tag{14}$$

By introducing the similarity variable, $\eta = y\sqrt{\dfrac{U_\infty}{2\cdot v \cdot x}}$, with $u = U_\infty \cdot f'(\eta)$ and $\phi(\eta) = \dfrac{T - T_\infty}{U_\infty^2/2\cdot c_p}$, the laminar boundary-layer equations with viscous dissipation reduce to

$$\begin{aligned} &f\cdot f'' + f''' = 0 \\ &\phi'' + \mathrm{Pr}\cdot f \cdot \phi' + 2\cdot \mathrm{Pr}\cdot(f'')^2 = 0 \end{aligned} \tag{15}$$

with the boundary conditions for adiabatic flow: $\eta = 0$, $f(0) = f'(0) = \phi'(0) = 0$ and $\eta \to \infty$, $f'(\infty) = 1$ and $\phi(\infty) = 0$. In the same fashion as for laminar boundary layers, the similarity equations for boundary-layer flow with viscous dissipation are not coupled and can be solved in the same manner as the laminar boundary-layer flows. Figure 7 provides the Mathcad worksheet for the solution for laminar boundary-layer flow with viscous dissipation. The momentum equation is solved as in Figure 2, and its solution is not repeated in Figure 7, which examines only the energy equation solution with viscous dissipation. The Mathcad procedure for laminar boundary-layer flow with viscous dissipation is similar to that already discussed for laminar boundary-layer flow with no dissipation. The boundary condition, $\phi'(0) = 0$, specifies a zero temperature gradient at the wall; thus the solution returns the adiabatic wall temperature—the wall temperature for zero heat flux. For dissipative flows, the adiabatic wall temperature is an important parameter and concept as is the recovery factor, r.

Consider the definition of the dimensionless temperature, ϕ, solved for the zero heat flux boundary condition. The result is

$$T_{aw} = T_{\infty+} + \phi(0)\cdot\frac{U_\infty^2}{2\cdot c_p} \tag{16}$$

where $\phi(0)$ is defined as the recovery factor, r, and T_{aw} is defined as the temperature an insulated wall would attain in a flow with viscous dissipation. The Mathcad computed results for the recovery factor agree well with those of White (1991), and for laminar flow in the range $0.1 < \mathrm{Pr} < 20$, r is approximately $\sqrt{\mathrm{Pr}}$.

Ancillary Mathcad Applications

The arithmetic/plotting capabilities of Mathcad are very useful for amplifying many concepts and convective heat transfer concepts are no exception. In a first course in convective heat transfer, especially for students with little background in viscous and turbulent flow, the language and formulation of turbulence can be daunting. Mathcad can help demonstrate many concepts and easily and quickly answer "what-if" questions.

The simplest, viable approach to closure is the Prandtl mixing length formulation as embodied in the van Driest mixing length expression (Bejan). Consistent with the van Driest mixing length formulation is a differential equation representing the viscous sublayer and overlap regions of a turbulent boundary layer. Expressed in inner region coordinates, $u^+ = \dfrac{\bar{u}}{u_*}$ and $y^+ = \dfrac{y \cdot u_*}{\nu}$ where $u_* = \sqrt{\dfrac{\tau_w}{\rho}}$ is the friction velocity, the van Driest representation is

$$\frac{du^+}{dy^+} = \frac{2}{1 + \left[1 + 4 \cdot \kappa^2 \cdot \left(y^+ \right)^2 * (1 - \exp(-\frac{y^+}{A^+}))^2 \right]^{0.5}} \tag{17}$$

where $\kappa = 0.40$ and $A^+ = 26$ for zero-pressure gradient boundary layers. A turbulent boundary layer is often discussed using the viscous sublayer, $u^+ = y^+$ for $0 \le y^+ \le \cong 10$, and the overlap region, $u^+ = \dfrac{1}{\kappa} \cdot \log(y^+) + B$ for $30 \le y^+ \le \cong 300$, where B = 5.5. Additionally, much is made of "power-law" profiles such as $u^+ = 8.7 \cdot \left(y^+ \right)^{\frac{1}{7}}$. Students often find the ideas expressed in myriads of turbulent boundary-layer profiles difficult to sort out and difficult to assimilate into a comprehensive understanding of the turbulent boundary layer. Mathcad can provide useful arithmetic aids.

Figure 8 provides examples of how Mathcad can be used to further understanding of turbulent flow concepts. The first part of Figure 8 utilizes RKFIXED to obtain the velocity profile corresponding to Equation (17). The results of the integration when plotted with the viscous sublayer and overlap region formulations and the power-law profile, reveals that the van Driest formulation faithfully tracks both the viscous sublayer and the overlap region and in a smooth, continuous fashion bridges the gap between the two regions. The power-law profile is reasonable in the overlap region, but not valid in the viscous sublayer. Mathcad can be used to generate other profile plots that are useful in helping students assimilate turbulent flow concepts.

Pedagogical Inferences

The procedure and examples presented in the previous section have been utilized in the Mississippi State University Mechanical Engineering graduate course, ME 8333 Convective Heat Transfer. All classical laminar convective heat transfer problems were solved using the techniques presented herein. Some solutions were not presented in class, but were assigned as homework. Observation by the instructor and conversations with the students indicate an improved appreciation of boundary-value problems, more attention given to the solution values (including the values of the "missing" boundary conditions), an enhanced willingness to generate solutions to the classical laminar convective heat transfer differential equations, an appreciation of the historical context of pre-computer solutions of the classical problems, and increased familiarity with a useful engineering arithmetic tool (Mathcad). Additionally, many class examples and many homework assignments not involving differential equations, utilized the capabilities of Mathcad.

Mathcad provides the opportunity to extend and expand the problem selection available for homework over the traditional assignments in a convective heat transfer course. The solutions to many interesting homework problems in convective heat transfer require significant efforts in coding/debugging, thus limiting the number and scope that is usually assigned. Since Mathcad solutions are relatively easy to obtain, the number, scope, and variety of possible homework problems are increased without a corresponding student overload.

Results and Conclusions

The use of Mathcad in an introductory graduate-level convective heat transfer flow course was judged a success. Certainly the use of Mathcad did not revolutionize the order and content of this course, but Mathcad did enhance student understanding of the mathematics involved and student perceptions of generating solutions to convective heat transfer problems. The variety and complexity of homework problems assigned was also increased without a corresponding

increase in student workload. Mathcad was also found useful in presenting other aspects of the course, but the usage presented herein formed the bulk of the Mathcad applications.

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$$m(n) := \frac{n}{2} \cdot \left(3 - n^2\right) \qquad dmdn(n) := \frac{d}{dn} m(n) \qquad dmdn(0) = 1.5 \qquad a_1 := \left[\frac{2 \cdot dmdn(0)}{\int_0^1 m(n) \cdot (1 - m(n)) \, dn}\right]^{0.5} \qquad a_1 = 4.641$$

$$a_2 := \left[(2 \cdot dmdn(0)) \cdot \int_0^1 m(n) \cdot (1 - m(n)) \, dn\right]^{0.5} \qquad a_2 = 0.646 \qquad \int_0^1 m(p \cdot del) \cdot (1 - m(p)) \, dp \rightarrow \frac{-3}{280} \cdot del^3 + \frac{3}{20} \cdot del$$

$$Pr = 2 \cdot \frac{dmdn(0)}{\left(del \cdot a_1\right)^2 \cdot \left(\frac{3}{20} \cdot del\right)} \qquad value := \frac{2}{3 \cdot a_1} \cdot \sqrt[3]{45} \cdot \left(dmdn(0) \cdot a_1\right)^{\frac{1}{3}} \qquad value = 0.976 \qquad del = \frac{0.976}{Pr^{0.333}}$$

$$Nvalue := \frac{dmdn(0)}{value \cdot a_1} \qquad Nvalue = 0.331 \qquad Nu_x = 0.331 \cdot Re_x^{0.5} \cdot Pr^{0.333}$$

Figure 1. Mathcad Worksheet for the Integral Solution with the Cubic Profile

Momentum (Blasius) equation only.

$v_0 := 1$ Guess on missing initial condition, f"(0).

$$IC(x1, v) := \begin{pmatrix} 0 \\ 0 \\ v_0 \end{pmatrix}$$ Vector of initial conditions.

$$d(x, y) := \begin{pmatrix} y_1 \\ y_2 \\ -y_0 \cdot y_2 \end{pmatrix}$$ Definition of Blasius equation as three first-order DEs.

$test(x2, y) := y_1 - 1$ Convergence test (f' = 1 for large η) for missing initial conditions.

Call to **sbval** to determine f"(0).

$s := sbval(v, 0, 8, d, IC, test)$ $s = (0.4696)$ Value of missing initial condition.

Invoke **rkfixed** to solve the Blasius equation as well as δ^* and θ.

$$d(x, y) := \begin{bmatrix} y_1 \\ y_2 \\ -y_0 \cdot y_2 \\ 1 - y_1 \\ y_1 \cdot (1 - y_1) \end{bmatrix}$$ The Blasius equation.

δ^*

$\theta.$

$$IC := \begin{pmatrix} 0 \\ 0 \\ s_0 \\ 0 \\ 0 \end{pmatrix}$$ Define the initial conditions.

$z := rkfixed(IC, 0, 8, 80, d)$ Fixed-step Runge-Kutta solution.

The array, **z**, contains the results with the independent variable as the first column.

$i := 0, 1 .. 80$ range variable $\eta_i := \left(z^{\langle 0 \rangle}\right)_i$ η, the independent variable

$f_i := \left(z^{\langle 1 \rangle}\right)_i$ f(η) $fp_i := \left(z^{\langle 2 \rangle}\right)_i$ f'(η) $fpp_i := \left(z^{\langle 3 \rangle}\right)_i$ f"(η)
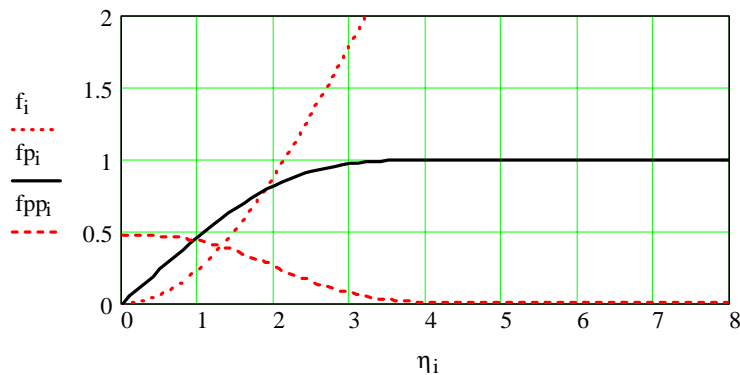

Figure 2. Mathcad Solution to the Momentum Equation

Momentum and energy equations.

$v_0 := 1.0$ Guess on missing initial condition, $\theta'(0)$.

$Pr := 100$ **Specify the Prandtl Number $= 100$.**

$$IC(x1, v) := \begin{pmatrix} 0 \\ 0 \\ s_0 \\ 0 \\ v_0 \end{pmatrix}$$ Vector of initial conditions.

$$d(x, y) := \begin{pmatrix} y_1 \\ y_2 \\ -y_0 \cdot y_2 \\ y_4 \\ -Pr \cdot y_0 \cdot y_4 \end{pmatrix}$$ Definition of equations.

$test(x2, y) := y_3 - 1$ Convergence test ($\theta = 1$ for large η) for missing initial condition.

Call to **sbval** to determine $\theta'(0)$.

$S := sbval(v, 0, 2, d, IC, test)$ $S = (2.2229)$ Value of missing initial condition.

Invoke **rkfixed** to solve the Blasius equation as well the energy equation.

$$IC := \begin{pmatrix} 0 \\ 0 \\ s_0 \\ 0 \\ S_0 \end{pmatrix}$$ Define the initial conditions.

$z := rkfixed(IC, 0, 8, 800, d)$ Fixed-step Runge-Kutta solution.

The array, **z**, contains the results with the independent variable as the first column.

$i := 0, 1 .. 800$ range variable $\eta_i := \left(z^{\langle 0 \rangle}\right)_i$ η, the independent variable

$f_i := \left(z^{\langle 1 \rangle}\right)_i$ $f(\eta)$ $fp_i := \left(z^{\langle 2 \rangle}\right)_i$ $f'(\eta)$

$fpp_i := \left(z^{\langle 3 \rangle}\right)_i$ $f''(\eta)$ $\theta_i := \left(z^{\langle 4 \rangle}\right)_i$ $\theta(\eta)$ $\theta p_i := \left(z^{\langle 5 \rangle}\right)_i$ $\theta p(\eta)$

Figure 3. Mathcad Solution to the Energy Equation
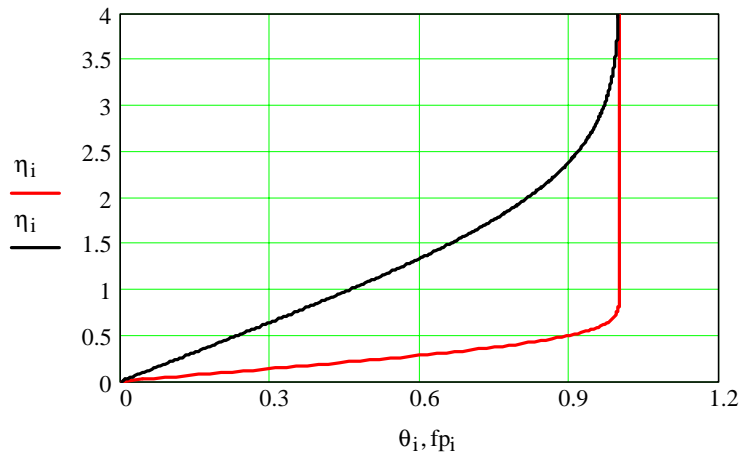
Temperature and velocity profiles:

Figure 3. Concluded

Uniform heat flux solution:

sbval used to determine the missing initial conditions for the differential energy equation.

$v_0 := 1.0$ Guess on missing initial condition, $\theta(0)$.

$$IC(x1, v) := \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$$ Vector of initial conditions.

$$d(r, \theta) := \begin{bmatrix} \theta_1 \\ \left[\left[\dfrac{\theta_1}{-r} - 2 \cdot \left(1 - r^2\right) \right] \text{ if } r > 0.0001 \\ \left[-2 \cdot \left(1 - r^2\right) \text{ otherwise} \right. \end{bmatrix}$$ Definition of energy equation as two first-order DEs with singularity removed.

$test(r2, \theta) := \theta_0$ Convergence test ($\theta = 0$ for $r = 1$) for missing initial conditions.

$s := sbval(v, 0, 1.0, d, IC, test)$ Call to **sbval** to determine $\theta(0)$. $s = (0.375)$

                                         Value of missing initial condition.

Figure 4. Mathcad Worksheet for Constant Heat Flux Duct Solution

$$d(r,\theta) := \begin{bmatrix} \theta_1 \\ \left\| \begin{bmatrix} \dfrac{\theta_1}{-r} - 2\cdot(1 - r^2) \end{bmatrix} \text{ if } r > 0.0001 \\ -2\cdot(1 - r^2) \text{ otherwise} \\ 4\cdot(1 - r^2)\cdot r\cdot\theta_0 \end{bmatrix}$$

θ_m definition

$$IC := \begin{pmatrix} s0 \\ 0 \\ 0 \end{pmatrix}$$

Define the initial conditions.

$z := \text{rkfixed}(IC, 0, 1, 40, d)$ Fixed-step Runge-Kutta solution.

$i := 0, 1 .. 40$ range variable

$r_i := \left(z^{\langle 0 \rangle}\right)_i$ r, the independent variable $\theta_i := \left(z^{\langle 1 \rangle}\right)_i$ $\theta(x)$

$\theta p_i := \left(z^{\langle 2 \rangle}\right)_i$ $\theta'(x)$ $\theta_m := \left(z^{\langle 3 \rangle}\right)_{\text{last}(z^{\langle 3 \rangle})}$ $\theta_m = 0.2291$ $Nu := \dfrac{1}{\theta_m}$ $Nu = 4.3656$

Examine the energy equation solution. θ' as a dimensionless temperature profile.
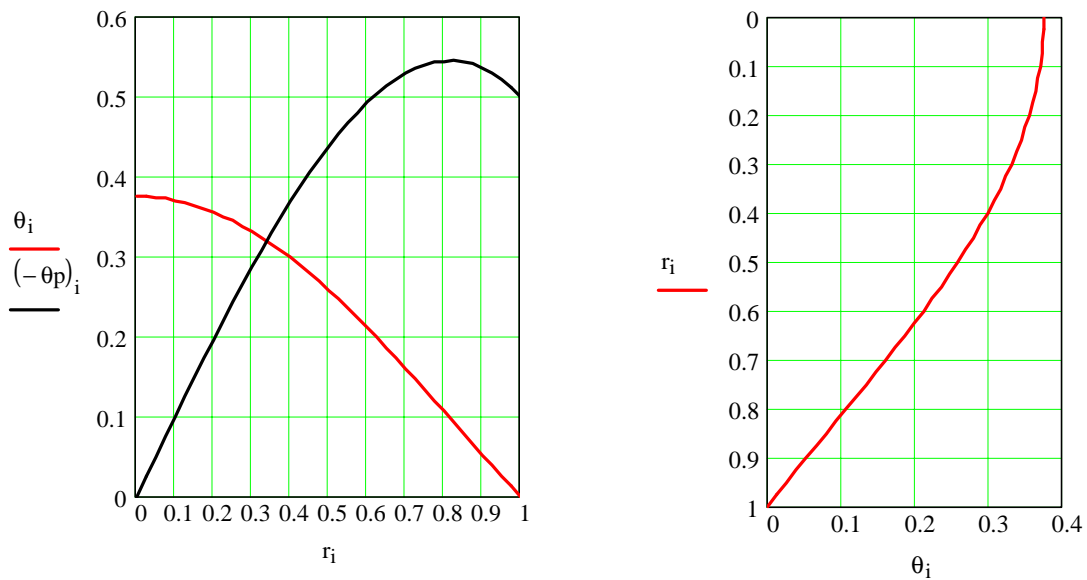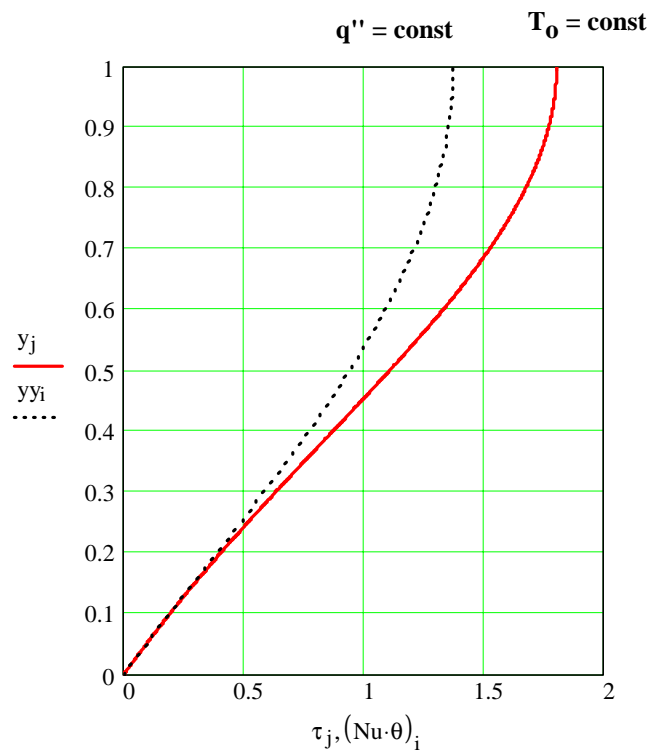

Figure 4. Concluded

Figure 5. Constant Heat Flux and Constant Surface Temperature Profiles

$$v := \begin{pmatrix} -0.46 \\ -1.05 \end{pmatrix}$$

Guess on missing initial conditions,

$\text{Pr} := 10$ **Specify the Prandtl number.**

$\theta'(0)$ and $F''(0)$.

$$\text{IC}(x1, v) := \begin{pmatrix} 1.0 \\ v_0 \\ 0.0 \\ 0.0 \\ v_1 \end{pmatrix}$$ Vector of initial conditions.

$$d(x, y) := \begin{bmatrix} y_1 \\ 0.75 \cdot y_2 \cdot y_1 \\ y_3 \\ y_4 \\ y_0 - \dfrac{1}{\text{Pr}} \cdot \left(0.5 \cdot y_3 \cdot y_3 - 0.75 \cdot y_2 \cdot y_4 \right) \end{bmatrix}$$

$$\text{test}(x2, y) := \begin{pmatrix} y_0 \\ y_3 \end{pmatrix}$$ Convergence test ($\theta = F' = 0$ for large η) for missing initial conditions.

Call to **sbval** to determine f''(0). $s := \text{sbval}(v, 0, 20, d, \text{IC}, \text{test})$ $s = \begin{pmatrix} -0.4649 \\ -1.0541 \end{pmatrix}$

Value of missing initial condition.

$$\text{IC} := \begin{pmatrix} 1.0 \\ s_0 \\ 0.0 \\ 0.0 \\ s_1 \end{pmatrix}$$ Define the initial conditions.

$z := \text{rkfixed}(\text{IC}, 0, 20, 2000, d)$ Invoke **rkfixed** to solve the energy and momentum equations.
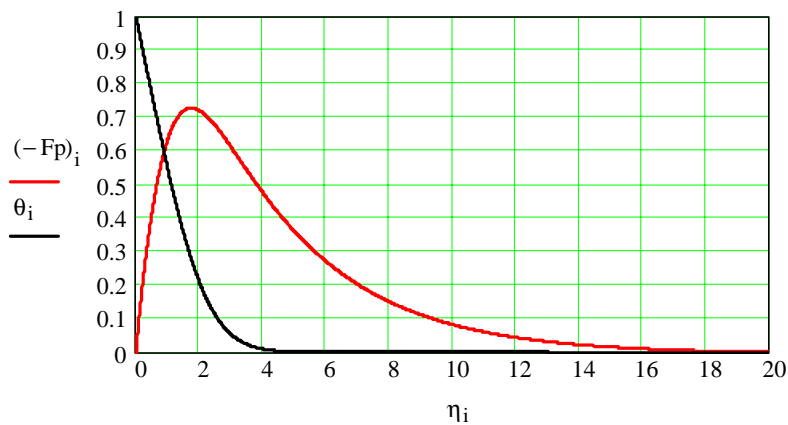

Figure 6. Laminar Free Convection Solution for Pr = 10

$Pr := 10$ **Specify the Prandtl number.**

$v_0 := (Pr)^{0.5}$ Guess on missing initial condition, $\phi(0)$.

$$IC(x1, v) := \begin{pmatrix} 0 \\ 0 \\ s_0 \\ v_0 \\ 0 \end{pmatrix}$$ Vector of initial conditions.

$$d(x, y) := \begin{bmatrix} y_1 \\ y_2 \\ -y_0 \cdot y_2 \\ y_4 \\ -Pr \cdot y_0 \cdot y_4 - 2 \cdot Pr \cdot (y_2)^2 \end{bmatrix}$$ Definition of equations.

$test(x2, y) := y_3$

Convergence test ($\phi = 0$ for $\eta = 0$) for missing initial condition.

Call to **sbval** to determine $\theta'(0)$.

     $S := sbval(v, 0, 6, d, IC, test)$ $S = (2.9616)$ Values of missing initial conditions.

Invoke **rkfixed** to solve the Blasius equation as well the energy equation.

$$IC := \begin{pmatrix} 0 \\ 0 \\ s_0 \\ S_0 \\ 0 \end{pmatrix}$$ Define the initial conditions.

$z := rkfixed(IC, 0, 8, 800, d)$
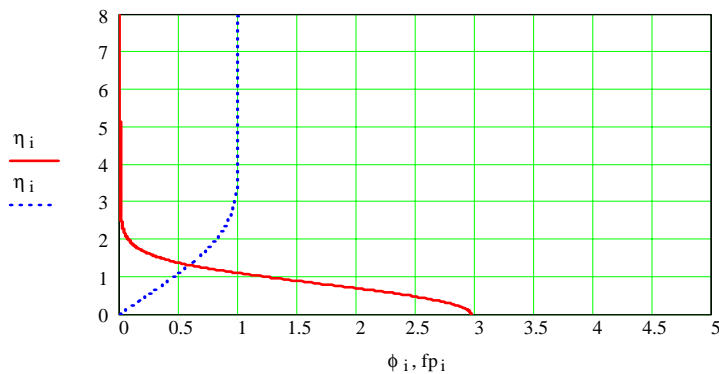
Fixed-step Runge-Kutta solution.

Figure 7. Boundary-layer Flow with Dissipation

$\kappa := 0.4$ von Karmen constant $Aplus := 26$ van Driest damping term

$$d(yplus, uplus) := \frac{2}{1 + \left[1 + 4 \cdot \kappa^2 \cdot yplus^2 \cdot \left(1 - \exp\left(\frac{-yplus}{Aplus}\right)\right)^2\right]^{0.5}}$$ van Driest differential equation

$IC := 0$ Initial condition on van Driest du^+/dy^+ differential equation

$z := rkfixed(IC, 0, 2000, 1000, d)$ Fixed-step Runge-Kutta solution.

The array, **z**, contains the results with the independent variable as the first column.

$i := 0, 1 .. 1000$ range variable $yplus_i := \left(z^{\langle 0 \rangle}\right)_i$ yplus, the independent variable

$uplus_i := \left(z^{\langle 1 \rangle}\right)_i$ uplus

Add the viscous sublayer and overlap region

viscous sublayer $j := 1 .. 15$ $yvs_j := j$ $uvs_j := yvs_j$

overlap region $jj := 5, 20 .. 2000$ $yo_{jj} := jj$ $uo_{jj} := \frac{1}{\kappa} \cdot \ln(yo_{jj}) + 5.5$

power law $jjj := 1, 10 .. 2000$ $ypl_{jjj} := jjj$ $upl_{jjj} := 8.7 \cdot \left(ypl_{jjj}\right)^{\frac{1}{7}}$
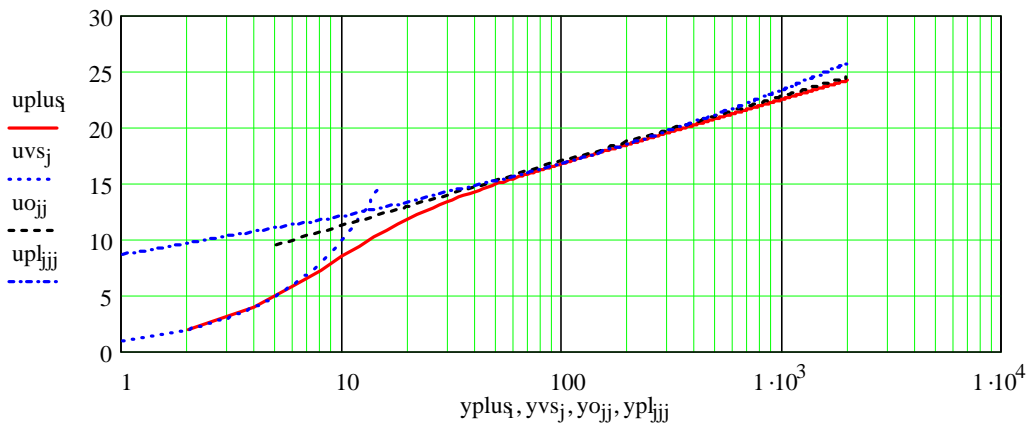

Figure 8. Examples of the Structure of the Turbulent Boundary Layer

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