Dynamic Effect of Prismatic Joint Inertia on Planar Kinematic Chains

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Abstract

The effect of prismatic joint inertia on dynamics of planar kinematic chains with friction is investigated. The influence of the prismatic joint inertia on the position of the application point of the joint reaction forces and on the dynamic response of a planar robot arm with feedback control is analyzed. Larger values of the initial condition response characteristics are observed for larger values of the slider link inertia. The numerical simulations reveal that the effect of slider inertia may be negligible at low speeds, but becomes significant at high speeds.

Introduction

Consideration of dynamic modeling is an important part in the analysis, design and control of mechanical systems such as mechanisms, robots, manipulators, etc. In general, mechanical systems have several desirable features relative to the coupling contact forces such as higher speed, improved mobility and control, and reduced power consumption. The dynamics of mechanical systems with frictional contacts has been developed and applied to many industrial applications. Examples in this area include fingered grippers [1] and manipulators [2]. The contact normal and tangential forces can be determined if the contacts are known for systems with independent constraints [3]. The contact forces cannot be uniquely determined when the constraints are not all independent. It has been shown that the initial value problem has no solution or multiple solutions for some initial conditions [4].
Do and Yang [5] solved the inverse dynamics of the Stewart platform manipulator [6] assuming the joints are frictionless and the moment of inertia of the legs has not been updated as a function of configuration in the simulation algorithm for path tracking. Ji [7] considered the question of leg inertia and studied its effect on the dynamics of the Stuart platform. The dynamic and gravity effects as well as the viscous friction at the joints were considered for the inverse dynamic formulation of the general Stuart platform presented by Dasgupta and Mruthyunjaya [8]. Important research related to the subject of the present paper has been done by Xi, Sinatra, and Han [9]. The authors investigated the effect of leg inertia on dynamic parameters of sliding-leg hexapods.
The theory presented in this study can be applied to the dynamic modeling of parallel manipulators with prismatic joints [10]. In general, the moment of inertia of prismatic joints is considered negligible for the dynamic analysis of kinematic chains. For high speed machine tools, the joint link inertia may become significant. In the present paper, the influence of the prismatic joint mass moment of inertia on dynamic parameters of mechanical systems as the application point of the joint contact forces, angular speed of the links, actuator torques and forces is analyzed. The results presented have direct application for the graduate courses in Mechanisms and Robotics and can help the students to better understand the dynamic concept of rigid body inertia.

Mathematical background

The Lagrange's equations of motion for the planar mechanical system shown in Fig. 1 are derived using constrained generalized coordinates. The cartesian reference frame $x_O O y_O$ is chosen. The mobile reference frame xOy attached to the link 1 is considered. The angle between the axis Ox and Ox_O is θ. For the links 1 and 2 the masses are m_1 and m_2, and the center of mass locations are designated by C_1 and C_2. The

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length of the link 1 is L. The distance OC_2 is denoted by r. The coefficient of friction between the links 1 and 2 is μ. The gravitational acceleration g is considered.
The gravitational forces $\mathbf{G}_1$ and $\mathbf{G}_2$ that act on the links 1 and 2 are

$$\mathbf{G}_1 = -m_1 g \left(\sin\theta \mathbf{\imath} + \cos\theta \mathbf{\jmath}\right), \; \mathbf{G}_2 = -m_2 g \left(\sin\theta \mathbf{\imath} + \cos\theta \mathbf{\jmath}\right). \tag{1}$$

The reaction force $\mathbf{F}_{12}$ and the friction force $\mathbf{F}_{f12}$ exerted by the link 1 to the link 2 can be written as

$$\mathbf{F}_{12} = N\mathbf{\jmath}, \; \mathbf{F}_{f12} = -\mu N \text{sign}(\dot{r})\mathbf{\imath}. \tag{2}$$

The reaction force $\mathbf{F}_{21}$ and the friction force $\mathbf{F}_{f21}$ exerted by the link 1 to the link 2 are

$$\mathbf{F}_{21} = -\mathbf{F}_{12}, \; \mathbf{F}_{f12} = -\mathbf{F}_{f12}. \tag{3}$$

The motion of the slider is expressed using the polar coordinates r and θ. To express the motion of the rod the angle ϕ is introduced. One can chose the generalized coordinates $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$.
The constraint equation is

$$\theta - \phi = 0. \tag{4}$$

The Lagrange differential equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \; i = 1, 2, 3, \tag{5}$$

where T is the total kinetic energy and Q_i is the generalized force corresponding to q_i.
The kinetic energy T_1 for the link 1 is

$$T_1 = \frac{1}{2} I_O \boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_1, \tag{6}$$

where $\boldsymbol{\omega}_1 = \dot{\phi}\mathbf{k}$.
The kinetic energy T_2 for the link 2 is

$$T_2 = \frac{1}{2} m_2 \mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2} + \frac{1}{2} I_{C_2} \boldsymbol{\omega}_2 \cdot \boldsymbol{\omega}_2, \tag{7}$$

where $\boldsymbol{\omega}_2 = \dot{\theta}\mathbf{k}$ and $\mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} + \boldsymbol{\omega}_2 \times \mathbf{r}_{C_2}$.
The total kinetic energy is

$$T = T_1 + T_2. \tag{8}$$

The velocity $\mathbf{v}_{P_1}$ of the point P attached to the link 1 can be written as

$$\mathbf{v}_{P_1} = \boldsymbol{\omega}_1 \times \mathbf{r}_{P_1}, \tag{9}$$

where $\mathbf{r}_{P_1} = p\left(\cos\phi \mathbf{\imath} + \sin\phi \mathbf{\jmath}\right)$.
The velocity $\mathbf{v}_{P_2}$ of the point P attached to the link 2 can be written as

$$\mathbf{v}_{P_2} = \boldsymbol{\omega}_2 \times \mathbf{r}_{P_2} + \dot{p}\mathbf{\imath}, \tag{10}$$

where $\mathbf{r}_{P_2} = p\left(\cos\theta \mathbf{\imath} + \sin\theta \mathbf{\jmath}\right)$.
The generalized force Q_i for the link i is

$$\begin{aligned} Q_i = {} & \frac{\partial \mathbf{v}_{C_1}}{\partial \dot{q}_i} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C_2}}{\partial q_i} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{P_1}}{\partial \dot{q}_i} \cdot (\mathbf{F}_{21} + \mathbf{F}_{f21}) + \\ & \frac{\partial \mathbf{v}_{P_2}}{\partial \dot{q}_i} \cdot (\mathbf{F}_{12} + \mathbf{F}_{f12}), \; i = 1, 2, 3. \end{aligned} \tag{11}$$

The sum of the moments for the link 2 with respect to C_2 is zero

$$(\mathbf{r}_{P_2} - \mathbf{r}_{C_2}) \times \mathbf{F}_{12} - I_{C_2} \boldsymbol{\alpha}_2 = \mathbf{0}, \tag{12}$$

where $\boldsymbol{\alpha}_2 = \ddot{\theta}\mathbf{k}$.
One can solve Eq. (12) and find p.

$$p = r + \frac{I_{C_2}}{N}\ddot{\theta}. \tag{13}$$

From Eq. (5), for $i = 3$, one can find the reaction force N. From Eq. (5), for $i = 1, 2$, and by using the constraint Eq. (4), one can derive and solve the equations of motion.

Robotic system

The equations of motion for the three-link planar robot arm (Fig. 2) are derived using Kane's method. The cartesian reference frame $x_O O y_O$ is chosen. The mobile reference frame $x_i O y_i$ is attached to the link i, $i = 1, 2$. The robot arm has three degrees of freedom corresponding to the angle q_1 between the link 1 and Ox-axis, the angle q_2 between the link 2 and Ox-axis, and the distance $q_3 = AC_3$. The links 1 and 2 have the lengths L_1 and L_2, the masses m_1 and m_2, and the mass moments of inertia I_{C_1} and I_{C_2}. The mass and the mass moment of inertia of the link 3 are m_3 and I_{C_3}. The coefficient of friction between the links 2 and 3 is μ. Friction is negligible for the rotational joints. The gravitational acceleration g is considered. The initial conditions at $t = 0$ are $q_1(0) = q_{10}$, $q_2(0) = q_{20}$, $q_3(0) = q_{30}$ m, and $\dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0$. The feedback control is implemented using the actuator torques $\mathbf{M}_{c01}$ and $\mathbf{M}_{c12}$ applied to the rotational joints O and A and the actuator force $\mathbf{F}_{c23}$ acting to the translational joint at A (Fig. 2). The desired final state of the system is $q_1 = q_{1f}$, $q_2 = q_{2f}$, and $q_3 = q_{3f}$.
The position $\mathbf{r}_{C_i}$ of the center of mass C_i, $i = 1, 2, 3$, is

$$\mathbf{r}_{C_1} = \frac{L_1}{2}\mathbf{\imath}_1,\ \mathbf{r}_{C_2} = \frac{L_2}{2}\mathbf{\imath}_2,\ \mathbf{r}_{C_3} = q_3\mathbf{\imath}_2. \tag{14}$$

The angular velocity $\boldsymbol{\omega}_i$ and angular acceleration $\boldsymbol{\alpha}_i$ of the link i is

$$\boldsymbol{\omega}_i = \dot{q}_i\mathbf{k}_i,\ \boldsymbol{\alpha}_i = \ddot{q}_i\mathbf{k}_i,\ i = 1, 2. \tag{15}$$

The velocities $\mathbf{v}_{C_i}$ and accelerations $\mathbf{a}_{C_i}$ of the points C_i can be expressed as

$$\mathbf{v}_{C_i} = \dot{\mathbf{r}}_{C_i} + \boldsymbol{\omega}_i \times \mathbf{r}_{C_i},\ \mathbf{a}_{C_i} = \dot{\mathbf{v}}_{C_i} + \boldsymbol{\omega}_i \times \mathbf{v}_{C_i}. \tag{16}$$

The reaction force $\mathbf{F}_{23}$ and the friction force $\mathbf{F}_{f23}$ exerted by the link 2 to the link 3 can be written as

$$\mathbf{F}_{23} = N\mathbf{\jmath}_2,\ \mathbf{F}_{f23} = -\mu N \mathrm{sign}(\dot{q}_3)\mathbf{\imath}_2. \tag{17}$$

The reaction force $\mathbf{F}_{32}$ and the friction force $\mathbf{F}_{f32}$ exerted by the link 1 to the link 2 are

$$\mathbf{F}_{32} = -\mathbf{F}_{23},\ \mathbf{F}_{f32} = -\mathbf{F}_{f23}. \tag{18}$$

The feedback control of the arm is realized using the actuator torques and forces

$$\begin{aligned} M_{c01} &= -[c_1\dot{q}_1 + c2(q_1 - q_{10})] + \frac{L_1}{2}m_1 g\cos q_1 + \left(\frac{L_2}{2}m_2 + q_3 m_3\right) g\cos q_2, \\ M_{c12} &= -[c_3\dot{q}_2 + c_4(q_2 - q_{20})] + \left(\frac{L_2}{2}m_2 + q_3 m_3\right) g\cos q_2, \\ F_{c23} &= -[c_5\dot{q}_3 + c_6(q_3 - q_{30})] + m_3 g\sin q_2 + \mu N \mathrm{sign}(\dot{q}_3), \end{aligned} \tag{19}$$

where c_i, $i = 1, ..., 6$, are constants.
The position of the application point P of the reaction force $\mathbf{F}_{23}$ is

$$\mathbf{r}_P = p\mathbf{\imath}_2, \tag{20}$$

where $p = q_3 + \dfrac{I_{C_3}}{N}\ddot{q}_2$.
The velocities $\mathbf{v}_{P_2}$ and $\mathbf{v}_{P_3}$ of the point P attached to the links 2 and 3 can be written as

$$\mathbf{v}_{P_2} = \boldsymbol{\omega}_2 \times \mathbf{r}_P,\ \mathbf{v}_{P_3} = \boldsymbol{\omega}_2 \times \mathbf{r}_P + \dot{\mathbf{r}}_P. \tag{21}$$

The relative velocity $\mathbf{v}_{P_{32}}$ between the links 3 and 2 is

$$\mathbf{v}_{P_{23}} = \mathbf{v}_{P_3} - \mathbf{v}_{P_2} = \dot{\mathbf{r}}_P. \tag{22}$$

One can define the generalized speeds $u_i = \dot{q}_i$ corresponding to the generalized coordinates q_i, $i = 1, 2, 3$. To find the reaction force N one can introduce the generalized speed u_4 on the direction Ox_2 in the expression of the relative velocity $\mathbf{v}_{P_{23}}$

$$\mathbf{v}_{P_{23}} = \dot{\mathbf{r}}_P + u_4\mathbf{\jmath}_2. \tag{23}$$

Thus, the velocities $\mathbf{v}_{P_3}$ and $\mathbf{v}_{C_3}$ become

$$\begin{aligned} \mathbf{v}_{P_3} &= \boldsymbol{\omega}_2 \times \mathbf{r}_P + \dot{\mathbf{r}}_P + u_4 \mathbf{J}_2, \\ \mathbf{v}_{C_3} &= \boldsymbol{\omega}_2 \times \mathbf{r}_{C_3} + \dot{\mathbf{r}}_{C_3} + u_4 \mathbf{J}_2. \end{aligned} \tag{24}$$

The generalized forces Q_j associated to the generalized speeds u_j can be written as

$$\begin{aligned} Q_j = \frac{\partial \mathbf{v}_{C_1}}{\partial u_j} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C_2}}{\partial u_j} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{C_3}}{\partial u_j} \cdot \mathbf{G}_3 + \frac{\partial \boldsymbol{\omega}_1}{\partial u_j} \mathbf{M}_{c01} + \\ \frac{\partial (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1)}{\partial u_j} \mathbf{M}_{c12} + \frac{\partial \mathbf{v}_{P_{23}}}{\partial u_j} \cdot (\mathbf{F}_{23} + \mathbf{F}_{f23} + \mathbf{F}_{c23}),\ j = 1, 2, 3. \end{aligned} \tag{25}$$

The generalized inertia forces F_j can be written as

$$F_j = \sum_{i=1}^{3} \frac{\partial \mathbf{v}_{C_i}}{\partial u_j} \cdot (-m_i \mathbf{a}_{C_i}) + \sum_{i=1}^{3} \frac{\partial \boldsymbol{\omega}_i}{\partial u_j} \cdot (-I_{C_i} \boldsymbol{\alpha}_i),\ j = 1, 2, 3. \tag{26}$$

One can write Kane equations associated to the generalized speeds u_j as

$$F_j + Q_j = 0,\ j = 1, 2, 3. \tag{27}$$

From Eq. (27), for $j = 4$, one can find the reaction force N. From Eq. (27), for $j = 1, 2, 3$, one can derive and solve the equations of motion with respect to the generalized co-ordinates q_1, q_2, and q_3.

Results

Simulated results obtained from the three-link planar arm robot shown in Fig. 2 are presented. The lengths of the links 1 and 2 are $L_1 = L_2 = 0.1$ m. The masses of the links 1, 2, and 3 are $m_1 = m_2 = 1$ kg, and $m_3 = 0.2$ kg. The mass moments of inertia of the links 1 and 2 are $I_{C_1} = I_{C_2} = 0.01$ kgm^2. The coefficient of friction between the links 2 and 3 is $\mu = 0.5$. The gravitational acceleration is $g = 9.807$ m/s^2. The initial conditions at $t = 0$ are $q_1(0) = \pi/6$ rad, $q_2(0) = \pi/4$ rad, $q_3(0) = 0.01$ m, and $\dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0$. The feedback control is implemented using the constants $c_1 = c_2 = c_3 = c_4 = 0.1$, and $c_5 = c_6 = 1$. The desired final state of the system is $q_{1f} = q_{2f} = \pi/3$ rad, and $q_{3f} = 0.1$ m. The initial conditions response of the co-ordinate $q_2(t)$ for $I_{C_3} = 0$, $I_{C_3} = 0.05$ kgm^2, $I_{C_3} = 0.1$ kgm^2, and $I_{C_3} = 0.15$ kgm^2 is illustrated in Figures 6.a-d.

One can define the error $e_i(t) = q_i(t) - q_{if}$ for the co-ordinate q_i, $i = 1, 2, 3$. The maximum overshoot $e_{imax} = \max |e_i(t)|$ and the settling time t_{si} ($e_i(t) < e_{i_0}$ for $t > t_{s_i}$), can be computed for $i = 1, 2, 3$, where e_{i_0} is a constant. The maximum overshoot e_{2max} and the settling time t_{s_2} are computed for different values of I_{C_3}, where $e_{2_0} = 10^{-3}$ (Table 1). For $I_{C_3} = 0$, the maximum overshoot is approximately zero ($e_{2max} \approx 0$) and the settling time is $t_{s_2} = 4.83$ s. Larger values of e_{2max} and t_{s_2} are observed for larger values of I_{C_3} for the same control parameters values.

I_{C_3}[kgm^2]	0	0.05	0.1	0.15
e_{2max}	$\approx$0	0.02	0.049	0.068
t_{s_2}[s]	4.83	7.42	12.66	15.34

Table 1: The maximum overshoot e_{2max} and the settling time t_{s_2} computed for different values of I_{C_3}.

Conclusions

The effect of prismatic joint inertia on the dynamic parameters of planar kinematic chains is presented. The effect of the slider inertia may be negligible at low speeds but becomes significant at relatively high speeds. Dynamic response characteristics of a planar robot arm are compared for different values of the slider inertia.

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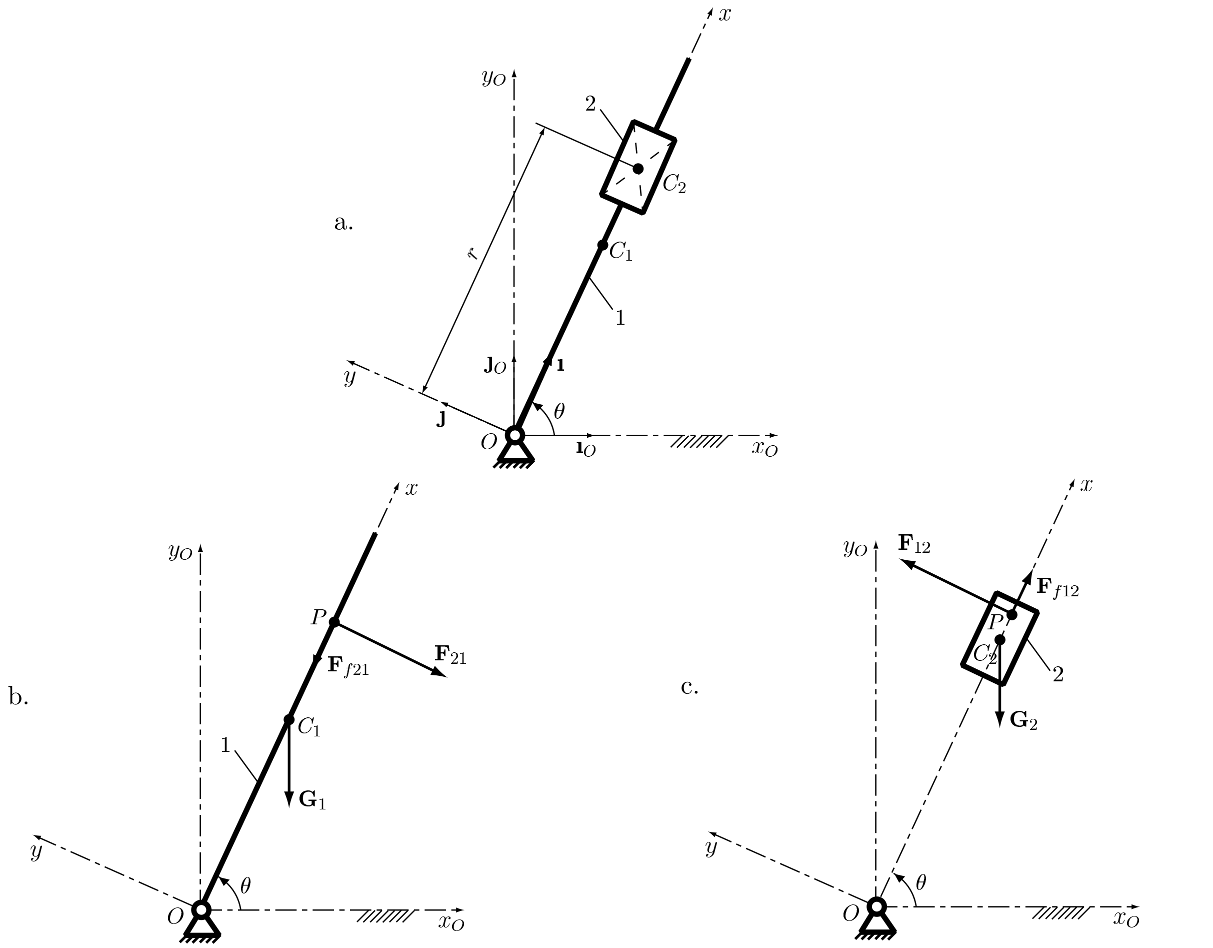

Figure 1: a. Open kinematic chain with slider and friction; b. Force diagram for the link 1; c. Force diagram for the link 2.

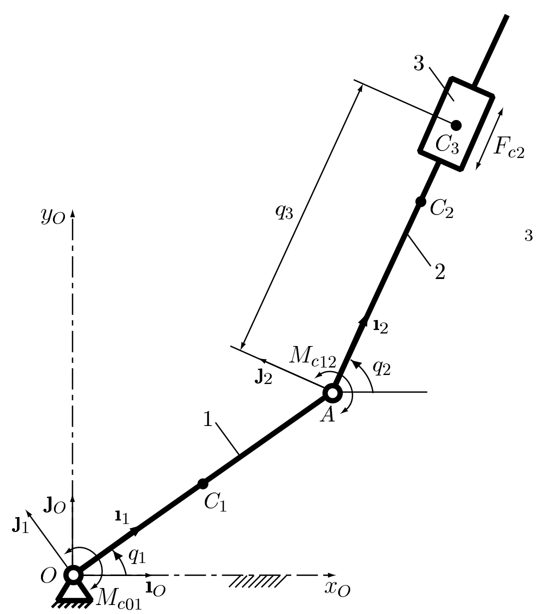

Figure 2: Three-link planar robot arm.

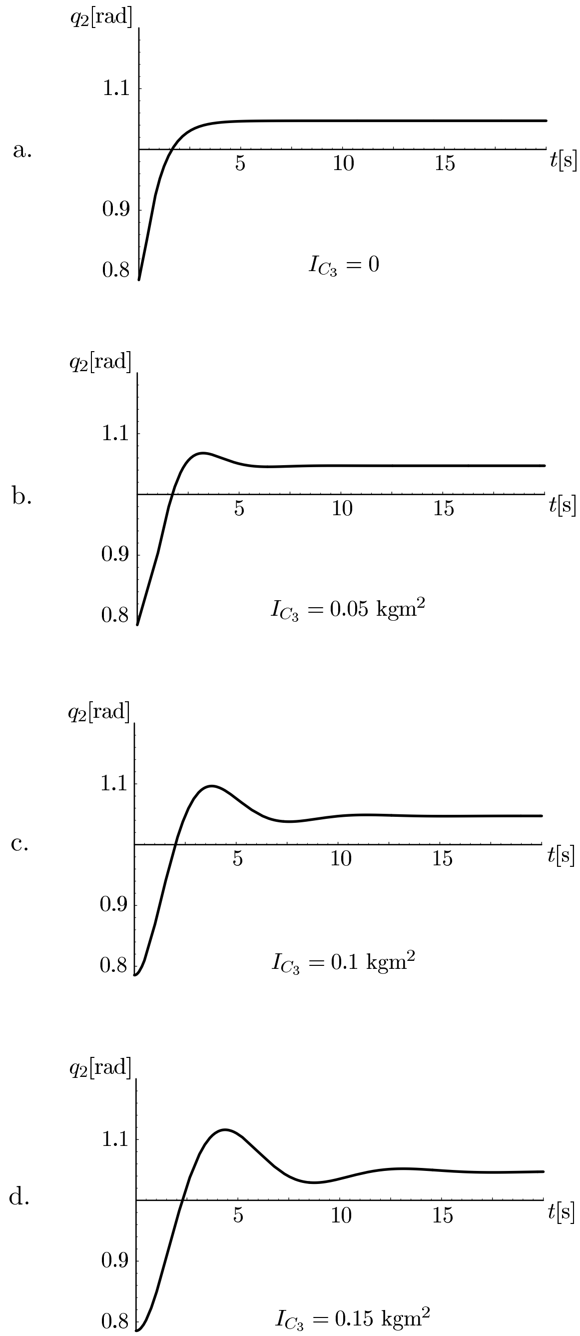

Figure 3: The dynamic response of the co-ordinate q_2 for the robot arm in the cases: a. $I_{C_3} = 0$, b. $I_{C_3} = 0.05$ kgm^2, c. $I_{C_3} = 0.1$ kgm^2, and d. $I_{C_3} = 0.15$ kgm^2.

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