Fundamental Shape Factors in Radiative Heat Transfer With Numerical Integration

P. S. Yeh, Ph.D.[1]

Abstract

To facilitate the numerical computation of shape factors in radiative heat transfer, a FORTRAN computer program has been developed. In comparison with the traditional method, which looks up numerical values from diagrams, the present computer program provides the shape factors as well as related information within a short time, the numerical values can be up to seven digits in accuracy.

Introduction

In the study of radiation heat transfer, in order to calculate the amount of heat transfer between two finite surfaces, the evaluation of the geometric shape factor between the two surfaces is the primary requirement. A shape factor is a measurement of the direct exchange of heat energy between two surfaces. The surfaces can be a rectangle, a circle, a cylinder, a cone, or some other basic geometric configurations. The surfaces can be placed in parallel, in perpendicular, or inclining to each other at a specified angle. The shape factor is determined by evaluating the exchange of heat energy between differential area elements in each surface and then performing simultaneous mathematical integration over both surfaces. If the Lambert's law of diffuse radiation is presumed to hold, the final expression is dependent solely on the geometry of the basic shapes involved.

For many combinations of geometric shapes, the mathematical expressions for the shape factors are rather complicated and lengthy [1], [2], [3]. Therefore, in practical application, they are traditionally represented in the form of a diagram, with appropriate parametric values to form a family of curves [1], [3], [4], [5], [6]. The accuracy of reading a numerical value from these diagrams is usually limited to about two numerical digits. Each time the geometrical dimensions are changed, new values of shape factors have to be obtained from these diagrams.

In the present study, a computer program written in FORTRAN has been developed. The program can be executed using a desktop computer with a FORTRAN compiler. Using a single-precision computation, the results can be up to seven digits in accuracy. Also, due to the fast increasing speed in electronic numerical processing, a lot of related data could be obtained in a matter of few minutes.

The computer program developed in the present study can be used to evaluate the following shape factors:

1. Between two finite, parallel, opposite rectangles of dimensions width, depth, and height
2. Between two perpendicular rectangles with a common edge (i.e., depth), of dimensions width, depth, and height
3. Between two finite rectangles with a common edge (i.e., depth), of dimensions width, depth, and slant, making an angle ϕ between the two surfaces
4. Between two parallel, concentric disks of radii r and R separated at a finite axial space L
5. Between two concentric cylinders of radii r and R of axial length L

In addition, shape factors for related geometric enclosures are also evaluated. These enclosures include a rectangular parallelepiped, a conical housing, and a cylindrical container.

[1] Professor of Engineering, Jacksonville State University, 700 Pelham Road North, Jacksonville, AL 36265

Mathematical Analysis

Notation

A_i	Area of surface i, m^2	s_4	=height/depth
$F_{i\text{-}j}$	Shape factor between surfaces i and j	s_5	=r/L
L	axial length, m	s_6	=R/L
r	radius of small disk or cylinder, m	s_7	=r/R
R	radius of large disk or cylinder, m	s_8	=L/R
s_1	=depth/height	s_9	=slant/depth
s_2	=width/height	s_{10}	=width/depth
s_3	=width/depth	ϕ	Inclined angle

1. The Shape Factor Between Two Finite, Parallel, Opposite Rectangles:

With the geometric arrangement shown in Figure 1, the shape factor $F_{1\text{-}2}$ between surfaces 1 and 2 is given by the following equation [3]:

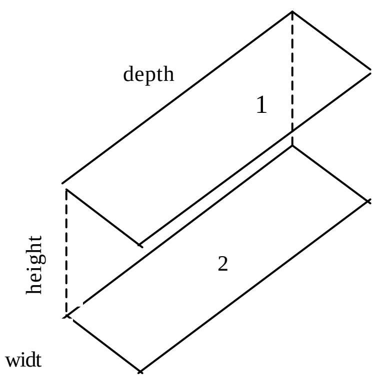

Figure 1. Parallel Surfaces

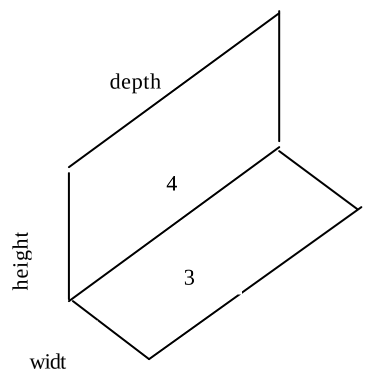

Figure 2. Perpendicular surfaces

$$F_{1-2}=\frac{1}{p}\{\frac{1}{s_1 s_2}\ln[\frac{(1+s_1^2)(1+s_2^2)}{(1+s_1^2+s_2^2)}]-\frac{2}{s_1}\tan^{-1}s_2-\frac{2}{s_2}\tan^{-1}s_1+2\sqrt{1+\frac{1}{s_1^2}}\tan^{-1}(\frac{s_2}{\sqrt{1+s_1^2}})+2\sqrt{1+\frac{1}{s_2^2}}\tan^{-1}(\frac{s_1}{\sqrt{1+s_2^2}})\} \tag{1}$$

2. The Shape Factor Between Two Perpendicular Rectangles With A Common Edge:

With the geometric arrangement shown in Figure 2, the shape factor $F_{3\text{-}4}$ between surfaces 3 and 4 is given by the following equation [3]:

$$\begin{aligned}F_{3-4} = {} & \frac{1}{p s_3}\langle s_3\tan^{-1}(\frac{1}{s_3})+s_4\tan^{-1}(\frac{1}{s_4})-\sqrt{s_3^2+s_4^2}\tan^{-1}(\frac{1}{\sqrt{s_3^2+s_4^2}}) \\ & +\frac{1}{4}\ln\{[\frac{(1+s_3^2)(1+s_4^2)}{(1+s_3^2+s_4^2)}]\cdot[\frac{s_4^2(1+s_3^2+s_4^2)}{(1+s_4^2)(s_3^2+s_4^2)}]^{s_4^2}\cdot[\frac{s_3^2(1+s_3^2+s_4^2)}{(1+s_3^2)(s_3^2+s_4^3)}]^{s_3^2}\}\rangle\end{aligned} \tag{2}$$

3. The Shape Factor Between Two Finite Rectangles With a Common Edge, Making an Angle ϕ Between Them:

With the geometric arrangement shown in Figure 3, in terms of the variables s_9=slant/depth, s_{10}=width/depth, and the angle ϕ, the shape factor F_{1-2} between surfaces 1 and 2 is given by the following equation [2]

$$
\begin{aligned}
(ps_{10})F_{1-2} = &-\frac{\sin 2f}{4}[s_9 s_{10}\sin f + (\frac{p}{2} - f)(s_9^2 + s_{10}^2) + s_{10}^2 \tan^{-1}(\frac{s_9 - s_{10}\cos f}{s_{10}\sin f}) + s_9^2\tan^{-1}(\frac{s_{10} - s_9\cos f}{s_9 \sin f})] \\
&+\frac{\sin^2 f}{4}\{(\frac{2}{\sin^2 f} - 1)\ln[\frac{(1+s_9^2)(1+s_{10}^2)}{1+z}] + s_{10}^2\ln[\frac{s_{10}^2(1+z)}{(1+s_{10}^2)z}] + s_9^2\ln[\frac{s_9^2(1+s_9^2)^{\cos 2f}}{z(1+z)^{\cos 2f}}]\} \\
&+ s_{10}\tan^{-1}(\frac{1}{s_{10}}) + s_9\tan^{-1}(\frac{1}{s_9}) - \sqrt{z}\tan^{-1}(\frac{1}{\sqrt{z}}) + \frac{\sin f\sin 2f}{2}s_9\sqrt{1+s_9^2\sin^2 f} \\
&\times[\tan^{-1}(\frac{s_9\cos f}{\sqrt{1+s_9^2\sin^2 f}}) + \tan^{-1}(\frac{s_{10} - s_9\cos f}{\sqrt{1+s_9^2\sin^2 f}})] + \cos f\int_0^{s_9}\sqrt{1+h^2\sin^2 f} \\
&\times[\tan^{-1}(\frac{s_9 - h\cos f}{\sqrt{1+h^2\sin^2 f}}) + \tan^{-1}(\frac{h\cos f}{\sqrt{1+h^2\sin^2 f}})]dh \qquad (3)
\end{aligned}
$$

Where $z = s_9^2 + s_{10}^2 - 2s_9 s_{10}\cos f$

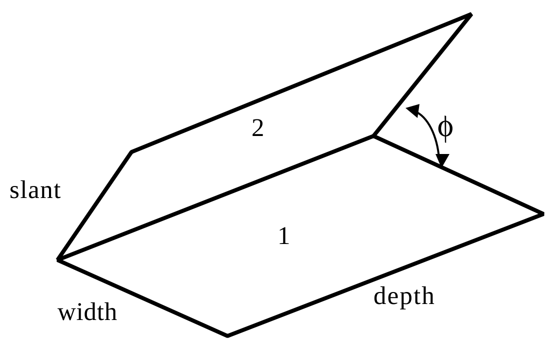

Figure 3. Two rectangular surfaces making an angle f

Note that for the same geometric configurations, when ϕ=90 degrees, the solution from Equation (3) should be the same as that from Equation (2).

4. The Shape Factor Between Two Parallel, Concentric Disks of Radii r and R, of Axial Length L:

With the geometric arrangement shown in Figure 4, in terms of the variables s_5=r/L and s_6=R/L, the shape factor F_{1-2} between the large disk R and the small disk r, at the axial length L, is given by the following equation [2]:

$$F_{1-2} = \frac{1}{2s_6^2}[1+s_5^2+s_6^2 - \sqrt{(1+s_5^2+s_6^2)^2 - 4s_5^2 s_6^2}] \qquad (4)$$

In the diagram, the conical surface shown with dashed lines is labeled with 3.

5. The Shape Factor Between Two Concentric Cylinders of Radii r and R, of Finite Axial Length L:

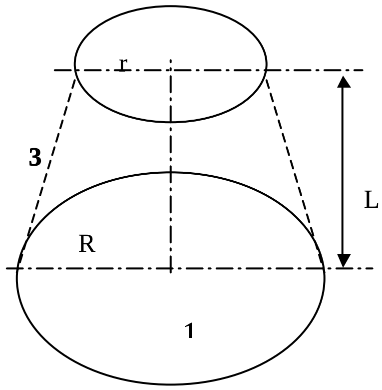

Figure 4. Parallel, Concentric Disks

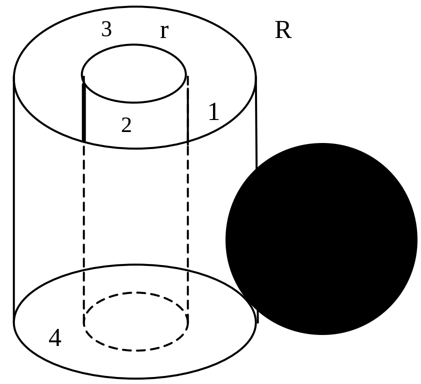

Figure 5. Concentric Cylinders

With the geometric arrangement shown in Figure 5, in terms of the variables s_7=r/R and s_8=L/R, the shape factors F_{1-1} and F_{1-2} are given respectively as follows [2]:

$$F_{1-1} = 1 + \frac{s_8}{4} - s_7 + \frac{2}{p} \cdot s_7 \tan^{-1}\left[2\sqrt{\frac{1}{s_8^2} - \left(\frac{s_7}{s_8}\right)^2}\right] + \frac{s_8}{2p}\sin^{-1}(1-2s_7^2) - \frac{\sqrt{4+s_8^2}}{2p}\left\{\frac{p}{2} + \sin^{-1}\left[1 - \frac{2s_7^2}{4\left(\frac{1}{s_8}\right)^2 - 4\left(\frac{s_7}{s_8}\right)^2 + 1}\right]\right\} \quad (5)$$

$$F_{1-2} = s_7\left[1 - \frac{1}{p}\cos^{-1}\left(\frac{s_7^2+s_8^2-1}{s_8^2-s_7^2+1}\right) + \frac{\sqrt{(s_7^2+s_8^2+1)^2-4s_7^2}}{2ps_8}\cos^{-1}\left(s_7\frac{s_7^2+s_8^2-1}{s_8^2-s_7^2+1}\right) + \frac{s_7^2+s_8^2-1}{2ps_8}\sin^{-1}(s_7) - \frac{s_8^2-s_7^2+1}{4s_8}\right] \quad (6)$$

In the diagram, the inside surface of the large cylinder is labeled as 1, the outside surface of the small cylinder labeled as 2, the top annular surface labeled as 3, and the bottom annular surface labeled as 4.

6. Shape Factor Properties

(A) The Reciprocity Property: Between any two surfaces A_i and A_j the product of the area and shape factor is equal. Hence

$$A_i F_{i-j} = A_j F_{j-i} \quad (7)$$

(B) The Additive Property: If a transmitting surface i is subdivided, the product of the original area and the shape factor with respect to a receiving surface j is the sum of product of the subdivided areas and their shape factors. Hence

$$A_i F_{i-j} = \sum_n A_{in} F_{in-j} \qquad with \quad A_i = \sum_n A_{in} \quad (8)$$

(C) The Sum of Shape Factors Property: The shape factor from a transmitting surface i to a subdivided receiving surface j is the sum of the individual, subdivided shape factors. Therefore,

$$F_{i-j} = \sum_n F_{i-jn} \qquad with \quad A_j = \sum_n A_{jn} \quad (9)$$

(D) The Enclosure Property: If a completely enclosed space consists of finite areas A_1, A_2, …, A_n, then n equations of the following form may be written for the shape factors:

$$\sum_{j=1}^{n} F_{i-j} = 1, \qquad i = 1,2,\ldots,n \quad (10)$$

Enclosure Analysis

A geometric enclosure can be created by proper combination of basic geometric shapes. For example, a rectangular parallelepiped consists of six rectangular surfaces, the surfaces are either parallel or perpendicular to each other. A conical enclosure consists of two parallel, concentric disks and a conical surface. A cylindrical enclosure consists of two concentric cylindrical surfaces with the two ends covered by circular rings. In conjunction with the various shape factor properties, the analyses of the basic shape factors between two surfaces can be extended to the study of an enclosure. Such a study is included in the computer program described below.

The Computer Program

The FORTRAN program consists of a main program and the following subprograms: (a) a subroutine for the evaluation of the shape factor between two parallel or perpendicular rectangles, and the analysis of a rectangular parallelepiped enclosure; (b) a subroutine for the evaluation of the shape factor between two rectangular surfaces with a common edge, making an angle ϕ between them. This subroutine refers to a FUNCTION subprogram which performs numerical integration, using the Simpson's rule, (c) a subroutine for the evaluation of the shape factor between two parallel concentric disks, and the analysis of a conical enclosure; (d) a subroutine for the evaluation of the shape factor between two concentric cylindrical surfaces, and the analysis of a cylindrical enclosure. A flowchart for the program is shown in Figure 6. In the program, the main program reads the input data. For the rectangular surfaces, the dimensions are specified by the height, depth and width. The value of angle is assigned as zero. For the case of two rectangular surfaces making an angle ϕ, the angle is given a value, with the height dimension replaced by slant. For the parallel disks or the cylindrical surfaces, the input values are the two radii and the axial length. The main program determines which subroutine is to be executed next. All computations and output are performed within the respective subroutine. Repeated execution of the entire program package can be done if multiple sets of input data are furnished.

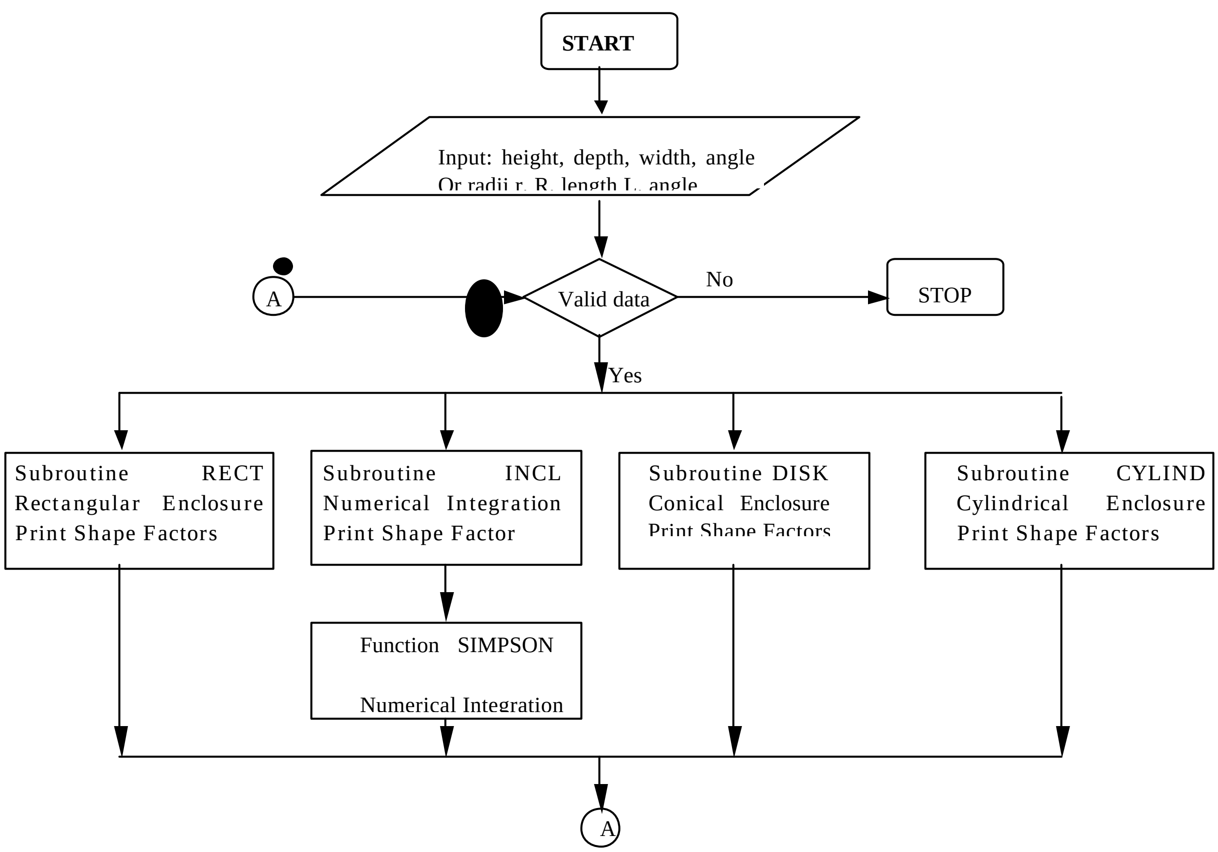

Figure 6. Flowchart for the Computer Program

Numerical Results

1. Rectangular Surfaces

A. Parallel Surfaces – Height=5.00 m (corresponds to the spacing between the two planes)
Width=10.00 m, Depth=20.00 m
Results: s_1=4.000 s_2=2.000 A_1=200.00 m^2 A_2=200.00 m^2 F_{1-2}=F_{2-1}=0.50899

B. Perpendicular Surfaces – Height=5.00 m, Width=10.00 m, Depth=20.00m
Results: s_3=0.500 s_4=0.250 A_1=200.00 m^2 A_2=100.00 m^2 F_{1-2}=0.16686
F_{2-1}=0.33371

C. Parallelepiped Enclosure – Same height, width, and depth as above. See Figure 7 for the surface identifications.

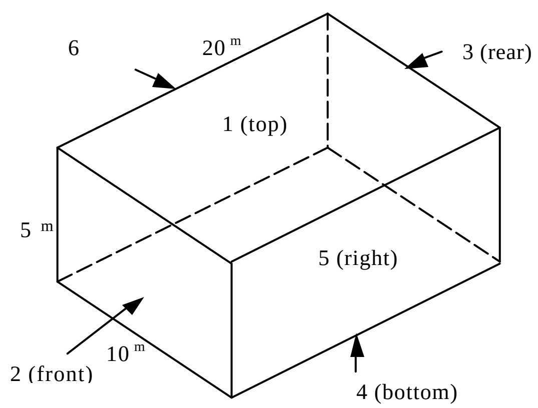

Figure 7. Surface Identification for a Parallelepiped Enclosure

A_1=200.00 m^2 (top surface) A_2=50.00 m^2 (front surface) A_5=100.00 m^2 (right-side)

Results: $F_{1-1} = F_{2-2} = F_{3-3} = F_{4-4} = F_{5-5} = F_{6-6} = 0.0000$
$F_{1-4} = F_{4-1} = 0.50899$ $F_{2-3} = F_{3-2} = 0.03618$ $F_{5-6} = F_{6-5} = 0.16527$
$F_{1-2} = F_{1-3} = F_{4-2} = F_{4-3} = 0.07865$ $F_{1-5} = F_{1-6} = F_{4-5} = F_{4-6} = 0.16686$
$F_{2-1} = F_{2-4} = F_{3-1} = F_{3-4} = 0.31460$ $F_{2-5} = F_{2-6} = F_{3-5} = F_{3-6} = 0.16731$
$F_{5-1} = F_{5-4} = F_{6-1} = F_{6-4} = 0.33371$ $F_{5-2} = F_{5-3} = F_{6-2} = F_{6-3} = 0.08365$

D. Two Rectangular Surfaces with a Common Edge, Making an Angle ϕ. See Figure 3 for the surface identification.

Slant=2.00 m, width=1.00 m, depth=20.00 m, angle=30^o.

Results: s_9=0.100 s_{10}=0.050 A_1=20.00 m^2 A_2=40.00 m^2 F_{1-2}=0.87041 F_{2-1}=0.43521

Slant=2.00 m, width=2.00 m, depth=20.00 m, angle=90^o.

Results: s_9=0.100 s_{10}=0.100 A_1=40.00 m^2 A_2=40.00 m^2 F_{1-2}=0.28189 F_{2-1}=0.28189

Note: These are the same results as would be obtained for two perpendicular rectangular surfaces.

2. Two Parallel Disks and Conical Enclosure

Using the same surface identifications as shown in Figure 4, with r=5.00 m, R=10.00 m, and L=10.00 m, the following results are obtained:

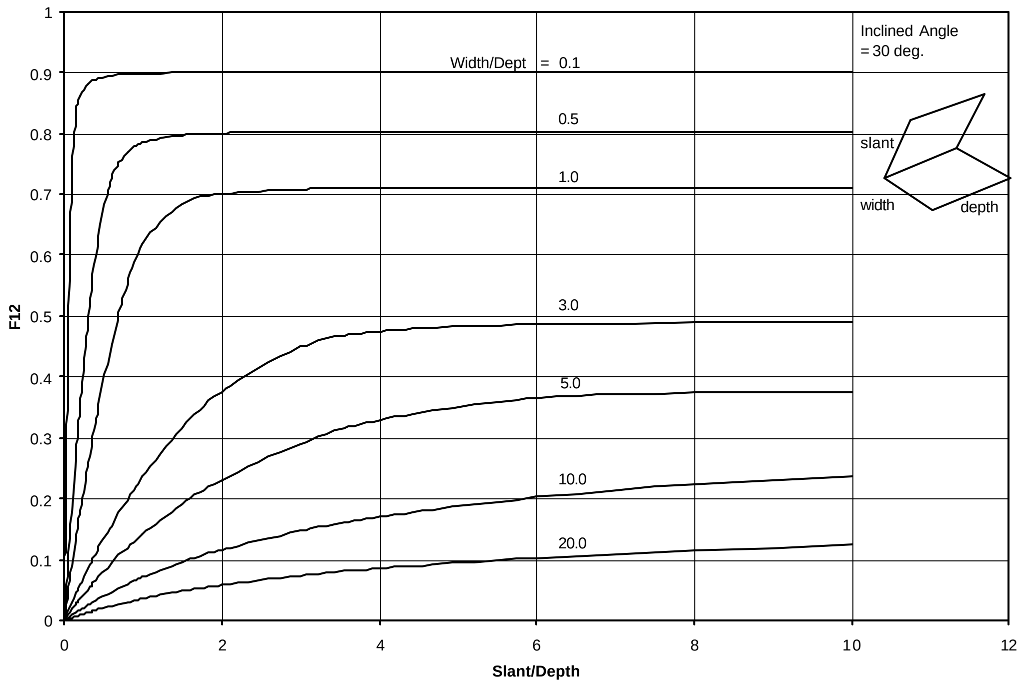

Figure 8. Shape Factor Between Two Rectangular Surfaces. ϕ = 30 deg.

A_1=314.16 m^2	A_2=78.54 m^2	A_3=526.86 m^2		
F_{1-1} = 0.00000	F_{1-2} = 0.11722	F_{1-3} = 0.88278	F_{2-1} = 0.46887	F_{2-2} = 0.00000
F_{2-3} = 0.53113	F_{3-1} = 0.52639	F_{3-2} = 0.07918	F_{3-3} = 0.39443	

3. Two Concentric Cylinders and Enclosure

Using the same surface identifications as shown in Figure 5, with r=0.40 m, R=1.00 m, and L=2.00 m, the following results are obtained:

A_1=12.5664 m^2	A_2=5.0265 m^2	A_3=2.6389 m^2	A_4=2.6389 m^2	
F_{1-1} = 0.38222	F_{1-2} = 0.31908	F_{1-3} = 0.14935	F_{1-4} = 0.14935	F_{2-1} = 0.79770
F_{2-2} = 0.00000	F_{2-3} = 0.10115	F_{2-4} = 0.10115	F_{3-1} = 0.71120	F_{3-2} = 0.19267
F_{3-3} = 0.00000	F_{3-4} = 0.09613	F_{4-1} = 0.71120	F_{4-2} = 0.19267	F_{4-3} = 0.09613
F_{4-4} = 0.00000				

4.. Two Rectangles With a Common Edge, Making an Angle ϕ Between Them

A sample numerical computer output is shown below, for an inclined angle of 30 degrees and width/depth=0.5.

Slant/Depth	0.0005	0.0025	0.0050	0.0100	0.0500	0.1000	0.1500	0.2000	0.3000	0.4000
F12	0.00093	0.00466	0.00930	0.01856	0.09119	0.17854	0.26200	0.34121	0.48404	0.59874

Slant/Depth	0.6000	0.8000	1.0000	1.5000	2.0000	3.0000	4.0000	6.0000	8.0000	10.0000
F12	0.72532	0.76790	0.78395	0.79667	0.80024	0.80242	0.80309	0.80352	0.80367	0.80373

Graphical presentations of the computer numerical results are shown in Figure 8 and Figure 9.

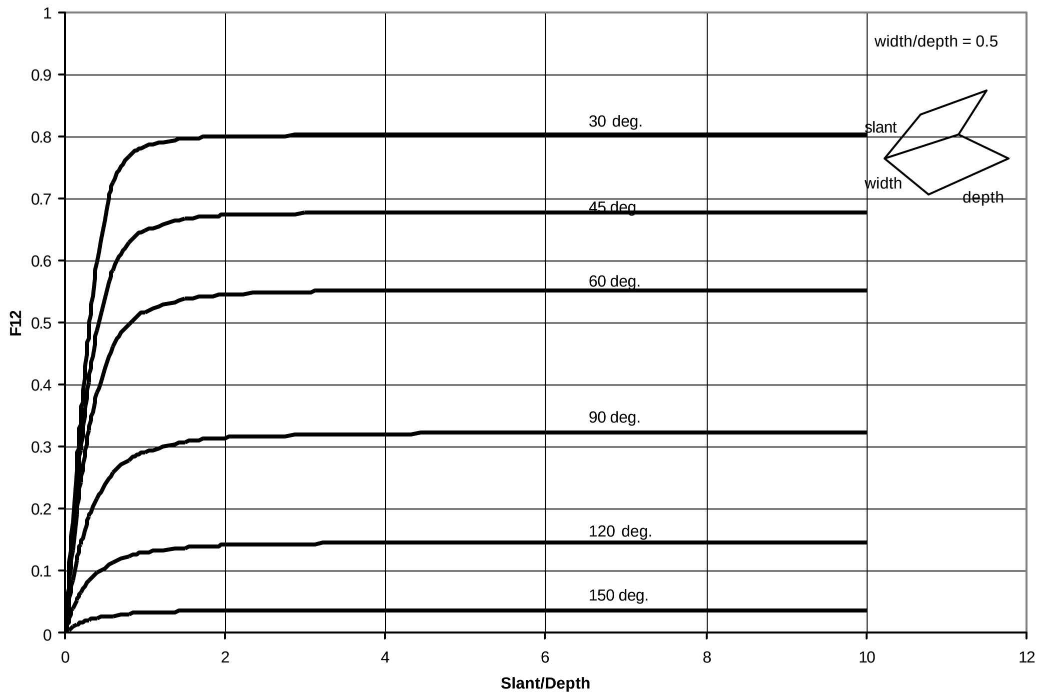

Figure 9. Shape Factor Between Two Rectangular Surfaces at Various Angles

Summary

A FORTRAN computer program has been developed for the study of radiation shape factors. The geometric configurations include: two parallel rectangles of the same size, two perpendicular rectangles with a common edge, two rectangular surfaces with a common edge making an angle ϕ, two parallel concentric disks, and two concentric cylinders of finite axial length. In comparison with the traditional method of using diagrams, the present computer program greatly facilitates the speed and accuracy of computation. The program can also be used to analyze related computation involving radiative heat transfer enclosure.

References

1. Chapman, Alan J. Fundamentals of Heat Transfer, 1987, pages 490-524. Macmillan Publishing Company, New York, N.Y.

2. Sparrow, E.M. and Cess, R.D. Radiation Heat Transfer, Revised Edition, 1970 Pages 113-135. Brooks/Cole Publishing Company, Belmont, CA.

3. Hottel, H.C., and Sarofim, A.F. Radiative Transfer, 1967, pages 50-54. McGraw-Hill Book Company, New York.

4. Holman, J.P. Heat Transfer, sixth edition, pages 373-400. McGraw-Hill Book Company. New York.

5. Lienhard, John H. A Heat Transfer Book, 1981,page 462. Prentice Hall, Inc. Englewood Cliffs, New Jersey.

6. Cengel, Yunus A. Introduction to Thermodynamics and Heat Transfer, 1997, pages 652-658. Irwin McGraw-Hill. New York, N.Y.

P. S. YEH, Ph.D.

P. S. Yeh received his B.S. degree from the National Taiwan University, in Taipei, Taiwan, the M.S. degree from the University of Illinois in Urbana-Champaign, Illinois, and the Ph.D. degree from Rutgers University in New Brunswick, New Jersey, all in Mechanical Engineering. He is a professor of Engineering at Jacksonville State University in Jacksonville, Alabama, where he has been teaching since 1967. His major areas of teaching and research are in fluid mechanics, thermodynamics, heat transfer, computer-aided design, computer programming, acoustics, and environmental engineering.