Getting Quick Right Answers
for
Hard Beam Problems

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Abstract

One of the most important and instructive classes of problems presented in a first course in strength of materials is the determination of stress and deformation for beams with concentrated loads and distributed loads with polynomial linear densities. The theoretical development is an excellent example of the type of reasoning found throughout the subject, and practical design problems using the results arise quite naturally. The traditional method of presentation of this subject develops the differential equations relating load, shear, and bending moment by analyzing a differential element from the beam under a distributed load only, and then integrates these equations between points at which concentrated loads are applied. This requires as many integrals per equation as there are concentrated loads and breaks in load distributions, burdening the analyst with the determination of the integrand for each of these integrals. The method presented here represents the entire load by a single linear distribution function. The concentrated loads appear as impulse functions, the distributed loads as step or (ramp)^n functions. The integration of the shear and moment equations then requires only one integral per equation. This is a considerably easier manual computation than the traditional procedure. It also provides a smooth path to computer solutions. Finally, the singularity function approach provides a vivid exposition of the relationships between mathematics, physics, computation, and engineering design practice.

Formulation

We begin by stating the equations relating load, shear, and moment for portions of beams subjected to distributed loads (see Figure 1):

$$\frac{dV}{dx} = -w(x) \tag{1}$$

$$\frac{dM}{dx} = V(x) \tag{2}$$

where: w(x) is the linear density of the distributed load with down as positive;

V(x) is the internal vertical force exerted *by* the right portion of the beam *on* the left portion of the beam; M(x) is the bending moment exerted *by* the right portion of the beam *on* the left portion of the beam.
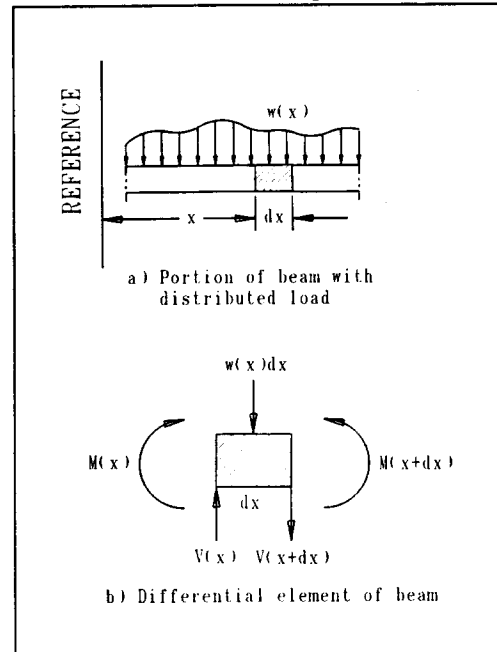

Figure 1. Beam with distributed load.

Concentrated Force Representation

Our next step is to develop a method of representing concentrated forces in terms of a linear distribution function so as to be able to express the entire problem in terms of a single w(x). I develop the impulse function along the usual
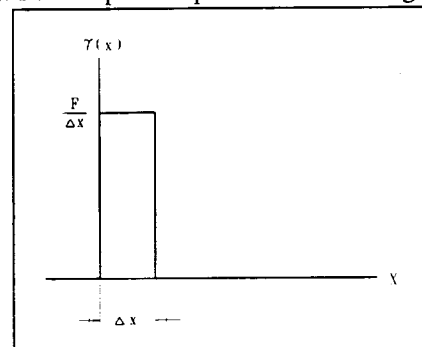

Figure 2. Impulse function development

heuristic lines, waving my hand at Figure 2., and pointing to equation (3):

$$F\delta(x) = \lim_{\Delta x \to 0} \gamma(x) \qquad (3)$$

After cautioning the class that the lack of rigor in this presentation would (rightly) give our math faculty apoplexy, I take this opportunity to expose the relationship between pure and applied mathematics, and to convey the importance of, and my respect for, each. I then make the argument that the area of the rectangular portion of $\gamma(x)$ is F, no matter what Δx is, so that

$$\int_{-\infty}^{x} F\delta(\alpha)d\alpha = 0, \; x < 0$$

$$\int_{-\infty}^{x} F\delta(\alpha)d\alpha = F, \; x \geq 0 \qquad (4)$$

The students have already seen that this is the V(x) function resulting from a concentrated force F applied at x=0. Then it is clear that w(x)=Fδ(x) is the load distribution function that represents the force F applied at x=0, and it is an easy demonstration that w(x)=Fδ(x-c) represents a force F applied at x=c. Now we can include concentrated forces in w(x). Our next step is to introduce compact expressions for successive integrals of impulse functions.

Step and (Ramp)^n Functions

The application of singularity functions to beams is presented with great clarity by Eisenberg[1]. The definitions used in the method being presented are expressed analytically in equations (5), and are shown graphically in Figure 3.

$$\text{Unit Step Function} = u(x) = \begin{cases} 0, \; x < 0 \\ 1, \; x \geq 0 \end{cases}$$

$$(\text{Unit Ramp Function})^n = \langle x \rangle^n = \begin{cases} 0, \; x < 0 \\ x^n, \; x \geq 0 \end{cases} \qquad (5)$$
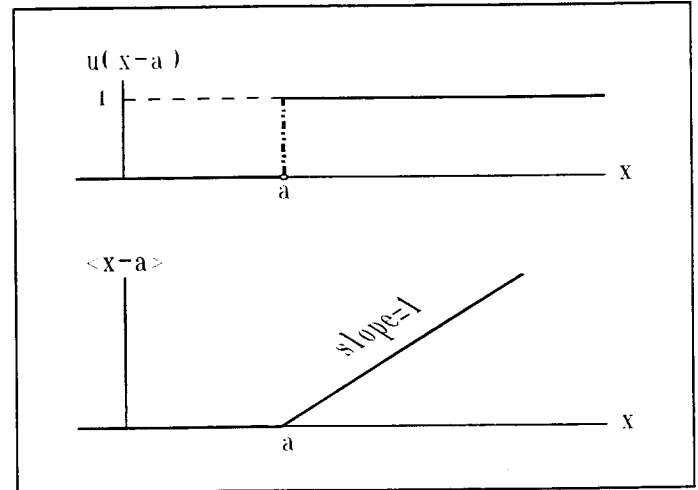

Figure 3. Unit Step and Unit Ramp Functions.

The rules of integration follow directly from (5):

$$\int_{0}^{x} \delta(\alpha - a)d\alpha = u(x - a)$$

$$\int_{0}^{x} u(\alpha - a)d(\alpha) = \langle x - a \rangle \qquad (6)$$

$$\int_{0}^{x} \langle \alpha - a \rangle^n d\alpha = \frac{1}{n+1} \langle x - a \rangle^{n+1}$$

Manual Computation Example

The usefulness of singularity functions in beam problems lies in these considerations:
1. Beam loads containing both concentrated forces and distributed loads represented by piecewise-polynomial linear density functions may be represented by a single linear density function consisting of the sum of impulse, step, and (ramp)^n functions. See Figure 4. for two examples.
2. The integration of equations (1) and (2) using (6) is very easy for this type of w(x).
3. The expansion of the singularity functions in the solution for the individual "piece" intervals also is very easy.

Figures 5. And 6. contrast the traditional and singularity function solutions for the shear and moment functions of a simply supported beam with one concentrated and one distributed load. The advantage in simplicity of the singularity function method apparent in this example becomes more pronounced as the loading becomes more complex and/or the computation of deflection is required. Another advantage of the singularity method is that the expansion of the shear and

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moment functions for values of x beyond the right end of the beam, which must be identically zero, may be done before the expansions for the other intervals. This provides a check before any more work is done.

a) $w(x) = [Wu(x-a) - Wu(x-b)]$ kip/in

b) $w(x) = [\dfrac{W}{b-a}\langle x-a \rangle + \dfrac{W}{b-a}\langle x-b \rangle - Wu(x-c)]$ kip/in
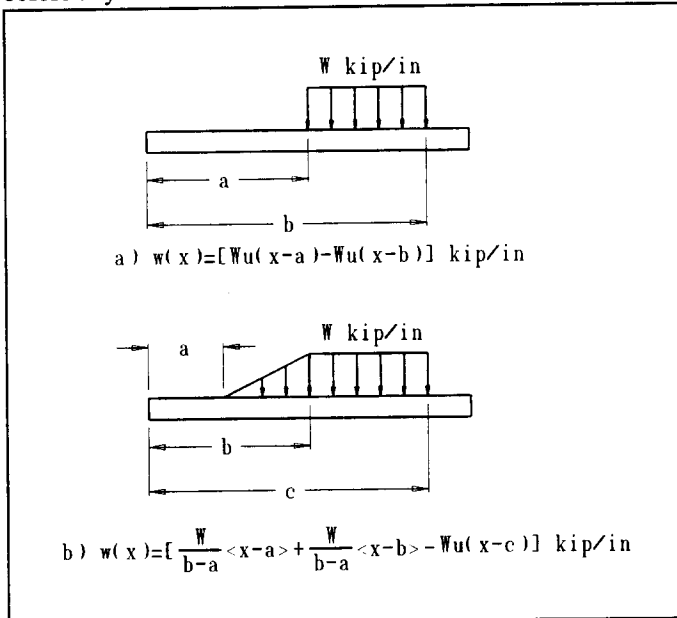
Figure 4. Linear density examples.

Maple Computation Example

Figure 7. shows the Maple computation, including Maple-generated shear and moment diagrams, of the singularity function formulation presented in Figure 6. Maple provides $\delta(x)$ and $u(x)$ as the built-in functions Dirac(x) and Heaviside(x) respectively. Lines two and three of Figure 7. serve to yield Maple output (line four) for $w(x)$ that looks exactly like what would be written manually. This facilitates checking. Lines five and seven also closely resemble the manual computation. Lines six and eight (which are not strictly necessary) do not closely resemble manual notation, but once the analyst learns to recognize "Heaviside(x)" as $u(x)$, he can recognize these expressions as their manual counterparts with the ramp functions expanded.

Design example

Figure 8. shows a design problem assigned as a take home test. A substantial number of trial cases involving beam loading calculations must be made in order to find the minimum weight of steel that will satisfy all three AISC load limitations. This very forcefully brings home to the students the importance of having the tools to get "quick" answers that we are confident are also "right."

Concluding Remarks

The approach to teaching beam computations described above typifies in several important ways the approach to engineering education that I personally advocate. First, carefully develop the application of basic physical principles to an engineering application. Second, couch the development in the most compact, economical mathematical form available, providing a heuristic development of the mathematical tools employed as required. Third, provide computational tools that translate the symbolic formulation of the application into correct numerical results as directly and naturally as possible. Finally, assign a *design* problem that brings home the practical value of this analytical approach.

Reference

1. Eisenberg, Martin A., Introduction to the Mechanics of Solids, Reading, Massachusetts, Addison-Wesley Publishing Company, 1980.

Linear dimensions in inches.

72 10 kip 0.1 kip/in

84   60

8.2 kip   180   7.8 kip

$0 \leq x < 72$

$\sum F = 0$

$8.2 - V(x) = 0$,

$V(x) = 8.2 \, kip$

$\sum M = 0$

$-x V(x) + M(x) = 0$

$M(x) = (8.2x) \, kip \cdot in$

$72 \leq x < 84$

10 kip

72   M(x)

x

8.2 kip   V(x)

$\sum F = 0$

$8.2 - 10 + V(x) = 0$

$V(x) = -1.8 \, kip$

$\sum M = 0$

$-(72)(10) - x V(x) + M(x) = 0$

$M(x) = (720 - 1.8x) \, kip \cdot in$

$84 \leq x < 144$

10 kip   .1 kip/in

72

84   M(x)

x

8.2 kip   V(x)

$\sum F = 0$

$8.2 - 10 - \int_{84}^{x} .1 \, dx - V(x) = 0$

$V(x) = -1.8 - .1x + 8.4$

$V(x) = (-.1x + 6.6) \, kip$

$\sum M = 0$

$-(72)(10) - \int_{84}^{x} x(.1) \, dx - x V(x)$

$M(x) = 0$

$M(x) = 720 + \frac{.1}{2}x^2 - \frac{.1}{2}(84^2)$

$- .1x^2 + 6.6x$

$M(x) = (-.05x^2 + 6.6x + 367) \, kip \cdot in$

$144 \leq x < 180$

10 kip   .1 kip/in

72

84   60   M(x)

x

8.2 kip   V(x)

$\sum F = 0$

$8.2 - 10 - .1(60) - V(x) = 0$

$V(x) = -7.8 \, kip$

$\sum M = 0$

$-(72)(10) - (84 + 30)(.1)(60)$

$- x V(x) + M(x) = 0$

$M(x) = (-7.8x + 1404) \, kip \cdot in$

Figure 5. Traditional Computation.

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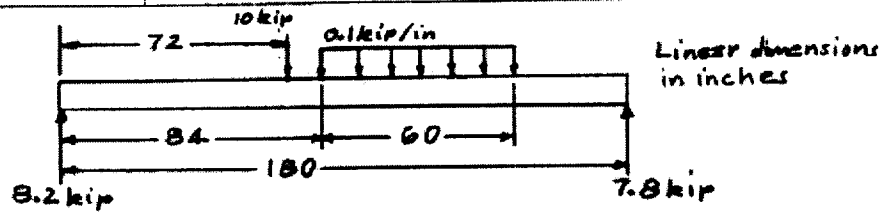
$$w(x) = -8.2\,\delta(x) + 10\,\delta(x-72) + .1\,\mathcal{U}(x-84) - .1\,\mathcal{U}(x-144) - 7.8\,\delta(x-180)$$

$$V(x) = \int_0^{180+} w(x) = 8.2\,\mathcal{U}(x) - 10\,\mathcal{U}(x-72) - .1\langle x-84\rangle + .1\langle x-144\rangle + 7.8\,\mathcal{U}(x-180)$$

$$M(x) = \int_0^{180+} V(x) = 8.2\langle x\rangle - 10\langle x-72\rangle - \frac{.1}{2}\langle x-84\rangle^2 + \frac{.1}{2}\langle x-144\rangle^2 + 7.8\langle x-180\rangle$$

$0 \leq x < 72$

$V(x) = 8.2\ kip$

$M(x) = (8.2x)\ kip\cdot in$

$72 \leq x < 84$

$V(x) = -1.8\ kip$

$M(x) = (-1.8x + 720)\ kip\cdot in$

$84 \leq x < 144$

$V(x) = (-.1x + 6.6)\ kip$

$M(x) = (-.05x^2 + 6.6x + 367)$
$\qquad\qquad kip\cdot in$

$144 \leq x < 180$

$V(x) = -7.8\ kip$

$M(x) = (-7.8x + 1404)\ kip\cdot in$

$180 \leq x$

$V(x) = 0$

$M(x) = 0$

Figure 6. Singularity function computation.

```
> Digits:=4;
```

$$Digits := 4$$

```
> delta:=x->Dirac(x);
```

$$\delta := Dirac$$

```
> u:=x->Heaviside(x);
```

$$u := Heaviside$$

```
> w:=x->-8.2*delta(x)+10*delta(x-72)+.1*u(x-84)-.1*u(x-144)-7.8*delta(x-180);
```

$$w := x \rightarrow -8.2\,\delta(x) + 10\,\delta(x-72) + .1\,u(x-84) - .1\,u(x-144) - 7.8\,\delta(x-180)$$

```
> V:=x->int(-w(x),x);
```

$$V := x \rightarrow \int -w(x)\,dx$$

```
> value(V(x));
```

$$8.200\ \text{Heaviside}(x) - 10.\ \text{Heaviside}(x-72.) - .1000\ \text{Heaviside}(x-84.)\,x + 8.400\ \text{Heaviside}(x-84.)$$
$$+ .1000\ \text{Heaviside}(x-144.)\,x - 14.40\ \text{Heaviside}(x-144.) + 7.800\ \text{Heaviside}(x-180.)$$

```
> plot(V(x),x=-1..181,axesfont=[TIMES,ROMAN,8],labels=[`        x in inches`,`V(x)`],title=`V(x) in
kips`,thickness=2,titlefont=[TIMES,BOLD,12] );
```

V(x) in kips


```
> M:=x->int(V(x),x);
```

$$M := x \rightarrow \int V(x)\,dx$$

```
> value(M(x));
```

$$8.200\ \text{Heaviside}(x)\,x - 10.\ \text{Heaviside}(x-72.)\,x + 720.\ \text{Heaviside}(x-72.) - .05000\ \text{Heaviside}(x-84.)\,x^2$$
$$- 352.8\ \text{Heaviside}(x-84.) + 8.400\ \text{Heaviside}(x-84.)\,x + .05000\ \text{Heaviside}(x-144.)\,x^2$$
$$+ 1037.\ \text{Heaviside}(x-144.) - 14.40\ \text{Heaviside}(x-144.)\,x + 7.800\ \text{Heaviside}(x-180.)\,x$$
$$- 1404.\ \text{Heaviside}(x-180.)$$

```
> plot(M(x),x=-.1..181,axesfont=[TIMES,ROMAN,8],labels=[`x in inches      `,`M(x)  `],title=`M(x) in
kip-inches`,thickness=2,titlefont=[TIMES,BOLD,12]);
```

M(x) in kip-inches

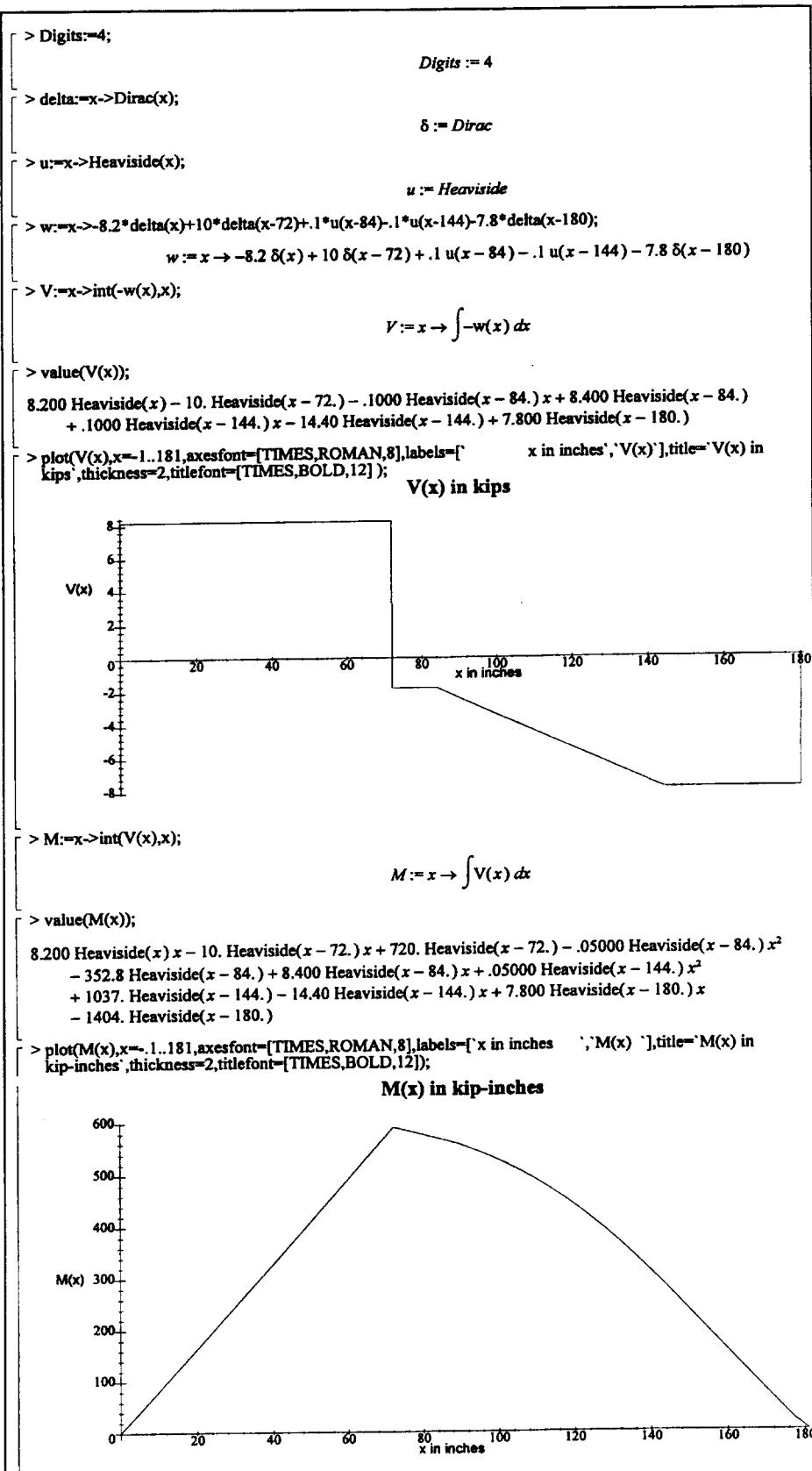*Figure 7. Maple computation.*

CODE REQUIREMENTS
Three of the specifications that the American Institute of Steel Construction places on the use of structural steel in beams are as follows.

a) The maximum tensile stress resulting from flexure must be less than 20,000 psi.

b) For beams without lateral bracing, compressive stress resulting from flexure must be less than the smaller of 20,000 psi and

$$\frac{22500}{1+\dfrac{L^2}{1800\,b^2}}\,psi$$

    where L is the unsupported length of the member
            b is the width of the compression flange.

c) For beams without lateral support, L must be no greater than 40b.

Rules b) and c) are to prevent lateral buckling.

PERFORMANCE SPECIFICATIONS
1. The floor configuration shown on the attached drawing must bear a load of 650 lb/ft^2 .
2. For adequate subfloor support, joists spacing must be no more than 9.5 ft between centers.

DESIGN SPECIFICATIONS
Select the girder span (17 ft or 18 ft), the W-shaped girder and the W-shaped joist that require the smallest total weight of steel.

DESIGN PRACTICE
1. Use the same members on the periphery and in the interior of the structure.
2. Treat each member as simply supported.

ASSIGNMENT
Generate the above DESIGN SPECIFICATIONS and present the calculations supporting them in an orderly and professional fashion.
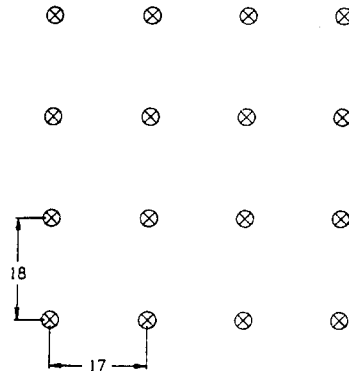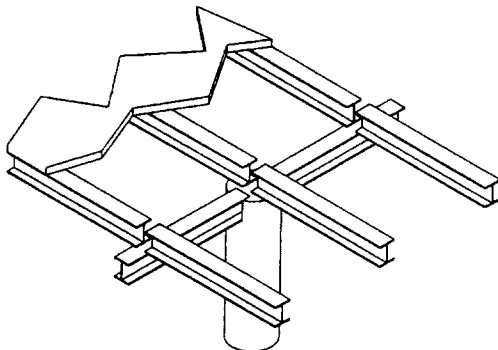
Dimensions in feet.
Dimensions typical.

Figure 8. Design problem.

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Dr. O'Connor received the B.E.E. degree from the University of Louisville in 1959. He earned the M.S.E.E. degree at the University of Southern California in 1961, and the Ph.D. degree at Purdue University in 1968. He was employed by the Hughes Aircraft Company at El Segundo, California from 1959 to 1964, where his work included control system design for the Surveyor spacecraft. He was employed by Arvin Industries, Columbus, Indiana, from 1964 to 1967, working principally on the advanced development of temperature compensated crystal oscillators. He served as Chief Executive Officer of Southland Electrical Supply Co., Louisville, Kentucky, from 1967 to 1992. In 1993, Dr. O'Connor joined the faculty of Lipscomb University, Nashville, Tennessee, where he currently is an Assistant Professor of Physics and Engineering Science.